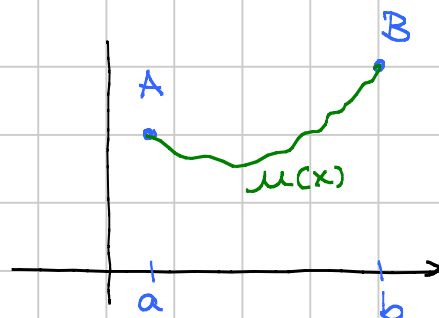


HEAVY CHAIN PROBLEM

Finding the shape of a chain or string subject to its own weight

UNEXTENDIBLE = FIXED LENGTH



Given (a, A) and (b, B) and $\ell > 0$

$$\min \left\{ \int_a^b u \sqrt{1 + \dot{u}^2} dx : \int_a^b \sqrt{1 + \dot{u}^2} dx = \ell \right\}$$

integral of u along the curve
= proportional to vertical coordinate of barycenter of the curve

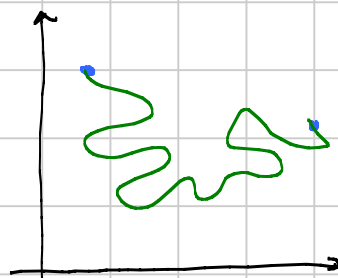
Above is the cartesian formulation. More generally we could consider the parametric approach

$$\gamma: [0, 1] \rightarrow \mathbb{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

$$\min \left\{ \int_0^1 y \sqrt{\dot{x}^2 + \dot{y}^2} dt : \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2} dt = \ell \right\}$$

$$\text{BC: } \begin{array}{ll} x(0) = a & y(0) = A \\ x(1) = b & y(1) = B \end{array}$$



Euler eqn with Lagrange multipliers
(cartesian formulation)

$$F(u) := \int_a^b u \sqrt{1 + \dot{u}^2} dx$$

$$G(u) := \int_a^b \sqrt{1 + \dot{u}^2} dx$$

→ Case $\delta G = 0$

→ Case $\delta(F + \lambda G) = 0$ for some $\lambda \in \mathbb{R}$

$$\boxed{\delta G = 0} \rightsquigarrow \left(\frac{\dot{u}}{\sqrt{1 + \dot{u}^2}} \right)' = 0 \rightsquigarrow \underbrace{\frac{\dot{u}}{\sqrt{1 + \dot{u}^2}} = c}_{\text{injective}} \rightsquigarrow \dot{u} = c_1$$

$\rightsquigarrow u$ is a line. $\rightsquigarrow \ell = |AB| \rightsquigarrow$ the line is actually the only competitor

$$\boxed{\delta(F + \lambda G) = 0} \quad L(s, p) := s \sqrt{1 + p^2} + \lambda \sqrt{1 + p^2} \\ = (s + \lambda) \sqrt{1 + p^2}$$

Up to λ , the Lagrangian is the same as in the min. surface of revolution. This implies that

$u(x) - \lambda$ is a stationary point for min surf. rev.
 $\Updownarrow$

$u(x)$ is a stationary point for the heavy chain.

It follows that our solutions are of the form

$$u(x) = c \cosh\left(\frac{x}{c} + k\right) + \lambda$$

depending on c, k, λ (and we have 2 BCs + length = ℓ)

System:

$$c \cosh\left(\frac{a}{c} + k\right) + \lambda = A$$

$$c \cosh\left(\frac{b}{c} + k\right) + \lambda = B$$

$$\boxed{\ell =} \int_a^b \sqrt{1 + \dot{u}^2} dx = \int_a^b \sqrt{1 + \sinh^2\left(\frac{x}{c} + k\right)} dx \\ = \int_a^b \cosh\left(\frac{x}{c} + k\right) dx = \left[c \sinh\left(\frac{x}{c} + k\right) \right]_a^b$$

Assume now that $[a, b] = [-1, 1]$ and set $\frac{\lambda}{c} =: t$.

$$c C(-t+k) - c C(t+k) = A - B$$

$$c S(t+k) - c S(-t+k) = \ell$$

$$\cancel{C(t)C(k)} - S(t)S(k) - \cancel{C(t)C(k)} - S(t)S(k) = \frac{A-B}{c}$$

$$S(t)C(k) + \cancel{C(t)S(k)} + S(t)C(k) - \cancel{C(t)S(k)} = \frac{\ell}{c}$$

$$S(t)S(k) = \frac{B-A}{2c}$$

$$S(t)C(k) = \frac{\ell}{2c}$$

Divide: $T(k) = \frac{B-A}{\ell} \stackrel{?}{\in} [-1, 1]$

↑
True 😊
because of geometry



~> we can find a unique k

$$C^2 - S^2 = 1$$

$$1 - T^2 = \frac{1}{C^2}$$

$$S(t) = \frac{\ell}{2c} \frac{1}{C(k)} = \frac{\ell}{2c} \sqrt{1 - T^2(k)}$$

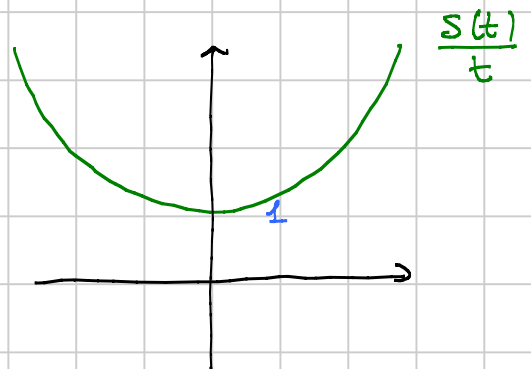
$$= \frac{\ell}{2c} \sqrt{1 - \left(\frac{B-A}{\ell}\right)^2}$$

$$S(t)^2 = \left(\frac{\ell}{2c}\right)^2 \left(1 - \left(\frac{B-A}{\ell}\right)^2\right)$$

$$= \left(\frac{\ell}{2\lambda}\right)^2 t^2 \left(1 - \left(\frac{B-A}{\ell}\right)^2\right)$$

$$\frac{\ell}{2c} = \frac{\ell \cancel{\lambda}^t}{2c\cancel{\lambda}} = \frac{\ell t}{2\lambda}$$

$$\left[\frac{S(t)}{t}\right]^2 = \underbrace{\left(\frac{\ell}{2\lambda}\right)^2 \left(1 - \left(\frac{B-A}{\ell}\right)^2\right)}_{\geq 1}$$



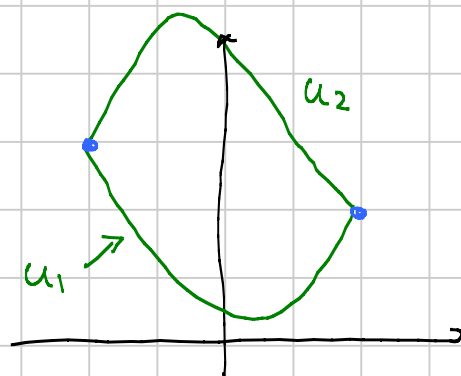
True $1 - \left(\frac{B-A}{\ell}\right)^2 \geq \left(\frac{2\lambda}{\ell}\right)^2 \Leftrightarrow \underbrace{(2\lambda)^2 + (B-A)^2}_{\text{distance AB}} \leq \ell^2$

Since conditions are satisfied, the system can be solved.
 Actually there are two solutions, one with $c > 0$ and one with $c < 0$.

They are "symmetric":

$u_1(x)$ is the GM $\rightarrow$ not proved

$u_2(x)$ is the global maximum



The result is proved if we can show that the min exist.

How to prove it?

Assume we want $\min \left\{ \int_a^b u (1 + \dot{u}^2) dx : \int \dot{u}^2 = 2 \right\}$

Can we apply direct method? More or less

$\rightarrow$ the constraint $\int \dot{u}^2 = 2$ gives compactness (+ BC)

$\rightarrow$ when we pass $\int_a^b \dot{u}_n^2 = 2$ to the limit we just obtain

that $\int_a^b \dot{u}_\infty^2 \leq 2$.

Fortunately

$$\min \left\{ \int \dots : \int \dot{u}^2 = 2 \right\} = \min \left\{ \int \dots : \underbrace{\int \dot{u}^2 \leq 2}_{\text{stable by weak convergence}} \right\}$$

In this case we say that the min SATURATES the constraint.

Alternative approach : reparam. the curves

$$\gamma: [0, \ell] \rightarrow \mathbb{R}^2$$

$$\dot{x}^2 + \dot{y}^2 = 1$$

$$\min \left\{ \int_0^\ell y \sqrt{\dot{x}^2 + \dot{y}^2} : \dot{x}^2 + \dot{y}^2 = 1 \right\} = \min \left\{ \int_0^\ell y \sqrt{\dot{x}^2 + \dot{y}^2} : \dot{x}^2 + \dot{y}^2 \leq 1 \right\}$$

stable by
weak convergence

The last prob. has a solution (by direct method)
and the solution saturates the condition.

