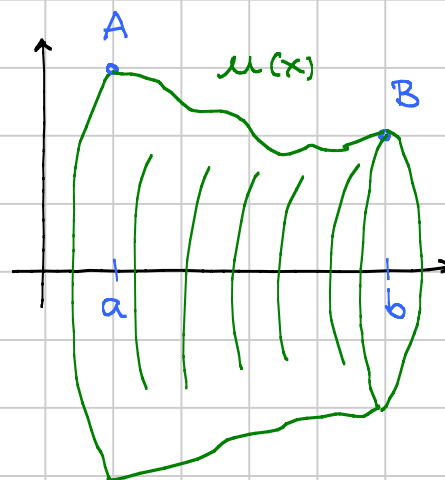


MINIMAL SURFACE OF REVOLUTION

$[a, b] \subseteq \mathbb{R}$ interval

A, B positive real numbers



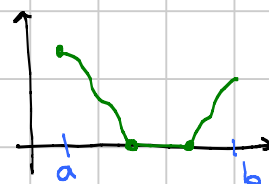
Finding $u(x)$ in such a way that the area of the revolution surface is min

$$\min \left\{ \underbrace{2\pi \int_a^b u \sqrt{1+u'^2} dx}_{\text{Area}} : u(a) = A, u(b) = B \right\}$$

Bad news: $L(s, p) := s \sqrt{1+p^2}$ has linear growth wrt p .

Remark We also assume that $u(x) \geq 0$ in $[a, b]$. This follows from minimality and usual truncation argument.

Euler equation Erdmann form



$$u L_p(u, u') = c + L(u, u')$$

$$u' \frac{u u'}{\sqrt{1+u'^2}} = c + u \sqrt{1+u'^2}$$

$$u \cancel{u'^2} = c \sqrt{1+u'^2} + u + u \cancel{u'^2}$$

$$\sqrt{1+u'^2} = \frac{u}{c}$$

$$u'^2 = \frac{u^2}{c^2} - 1$$

$$u' = \pm \sqrt{\frac{u^2}{c^2} - 1}$$

Brutal mode! $\int \frac{c du}{\sqrt{u^2 - c^2}} = \int \pm dx = \pm x + k$

$u = cv \rightsquigarrow du = c dv$ " $\int \frac{c \cdot c dv}{c \sqrt{v^2 - 1}} = c \arccos v = \pm x + k$

Therefore $\arccosh\left(\frac{u}{c}\right) = \frac{x}{c} + k$ so that finally

$$u(x) = c \cosh\left(\frac{x}{c} + k\right) \quad \text{with } c > 0, k \in \mathbb{R}$$

System (Courtesy Paolo Tilli) Finding c and k in order to fulfil BCs.

Let us assume wlog that $a = -r$, $b = r$ for some $r > 0$

$$u(-r) = c \cosh\left(-\frac{r}{c} + k\right) = A$$

$$u(r) = c \cosh\left(\frac{r}{c} + k\right) = B$$

$$\text{let us } t := \frac{r}{c} \quad C = \cosh \quad S = \sinh \quad T = \tanh$$

$$\cosh\left(-\frac{r}{c} + k\right) = \frac{A}{c} = \frac{A}{c} \frac{r}{r} = \frac{A}{c} t$$

$$\cosh\left(-\frac{r}{c} + k\right) = \frac{B}{c} \frac{r}{r} = \frac{B}{c} t$$

$$\text{so that} \quad C(-t+k) = \frac{A}{r} t \quad C(t+k) = \frac{B}{r} t$$

$$C(t)C(k) - S(t)S(k) = \frac{A}{r} t$$

$$C(t)C(k) + S(t)S(k) = \frac{B}{r} t$$

$$C(t)C(k) = \frac{B+A}{2r} t$$

$$S(t)S(k) = \frac{B-A}{2r} t$$

$$C(k) = \frac{B+A}{2r} \frac{t}{C(t)}$$

$$S(k) = \frac{B-A}{2r} \frac{t}{S(t)}$$

$$C(k)^2 - S(k)^2 = \left(\frac{B+A}{2r}\right)^2 \frac{t^2}{C(t)^2} - \left(\frac{B-A}{2r}\right)^2 \frac{t^2}{S(t)^2} = 1$$

$$1 + \left(\frac{B-A}{2r} \right)^2 \frac{t^2}{S(t)^2} = \left(\frac{B+A}{2r} \right)^2 \frac{t^2}{C(t)^2}$$

and finally

$$\boxed{\frac{C(t)^2}{t^2} + \left(\frac{B-A}{2r} \right)^2 \frac{1}{T(t)^2} = \left(\frac{B+A}{2r} \right)^2}$$

$\uparrow$
convex
 $\uparrow$
convex

Want: positive solutions of this equ.

Standard check: $\frac{C(t)}{t}$, $\frac{1}{T(t)}$ are convex nonnegative functions for $t > 0$

and the square is again convex.

Therefore we have 0/1/2 solutions depending on the RHS

Special case $A = B \rightsquigarrow k = 0$

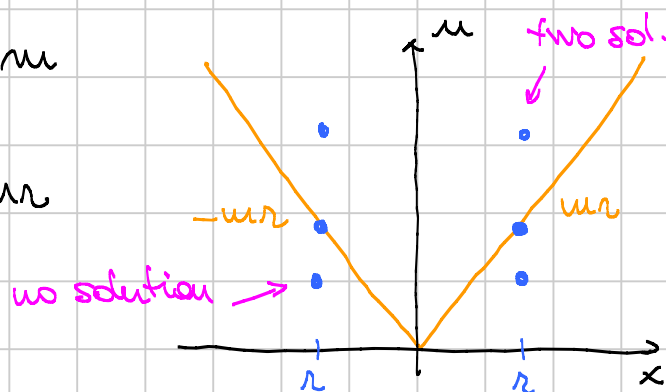
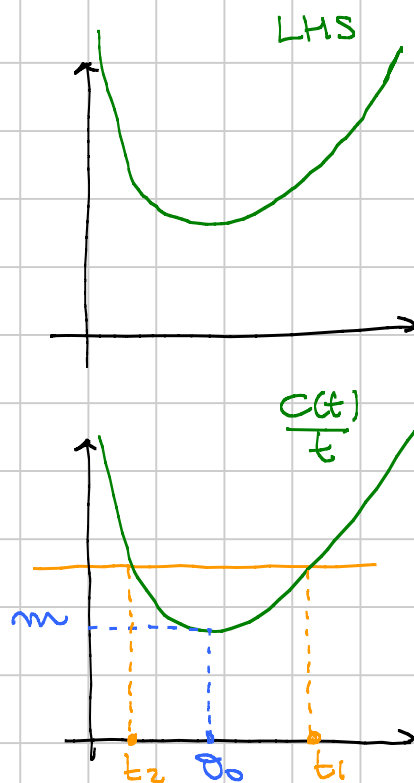
$$\frac{C(t)^2}{t^2} = \left(\frac{A}{r} \right)^2$$

$$\frac{C(t)}{t} = \frac{A}{r}$$

• No solution if $\frac{A}{r} < m$; $A < mr$

• One solution if $A = mr$

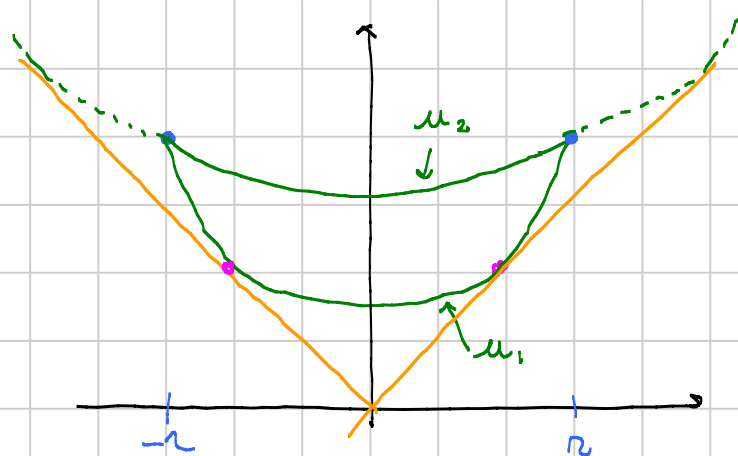
• Two solutions if $A > mr$



Case with two solutions

$$u_1(x) = c_1 \cos R \left(\frac{x}{c_1} \right) \quad c_1 < c_2$$

$$u_2(x) = c_2 \cos R \left(\frac{x}{c_2} \right)$$



Both solutions are tangent to $u|x|$, but u_1 is the solution of $[-r, r]$ and u_2 outside

For $x > 0$: $c \cos R \left(\frac{x}{c} \right) \geq mx$; $\frac{\cos R \left(\frac{x}{c} \right)}{\frac{x}{c}} \geq m$

with equality $\Leftrightarrow \frac{x}{c} = \theta_0 \Leftrightarrow x = c\theta_0$

We have to understand when $c\theta_0 > r \Leftrightarrow \frac{r}{c} < \theta_0$

Observe that t_1 and t_2 are on opposite sides of θ_0 .

FACTS

① When there are 2 solutions ($A > mr$) then

$u_2(x)$ is SLM

$u_1(x)$ is not even WLM

② If $A > mr$ and r is small enough, then $u_2(x)$ is GM

③ If $A < mr$, then the min. prob. has no solution

and the sup. is achieved by a discontinuous solution $\equiv 0$

(in some sense the solution are the two disks).

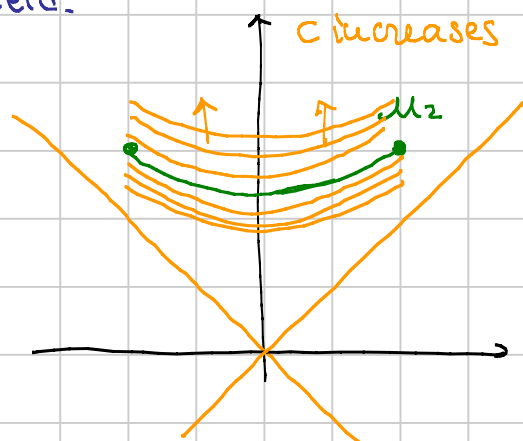


Proof That $u_2(x)$ is SLM $L(s,p)$ is convex in p
 $\Rightarrow$ excess ≥ 0 .

Enough to find a Weierstrass field.

$$u_2(x) = c \cosh\left(\frac{x}{c}\right)$$

As c varies this is a field.



Proof that u_2 is GM if ϵ is small

It is enough to prove that it is not convenient to exit the rectangle.

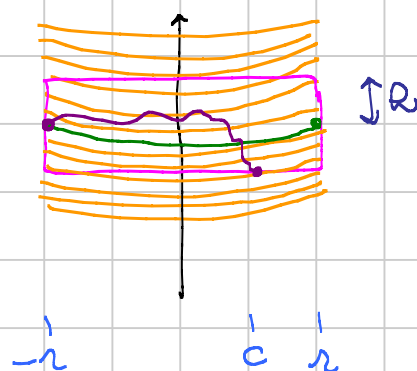
The exit cost can be estimated by

$$\int_a^b \dots \geq \int_a^c u \sqrt{1 + \tilde{u}^2} dx \geq \int u \tilde{u} \\ = \left[\frac{1}{2} u^2 \right]_a^c$$

$$= \frac{1}{2} \text{ something depending on } \epsilon$$

$$> F(u_2)$$

↑
true if ϵ is small enough



Sketch of why u_1 is not DLM

this is equivalent
 to saying that
 sol. to Jacobi equ
 do not have
 constant sign.

