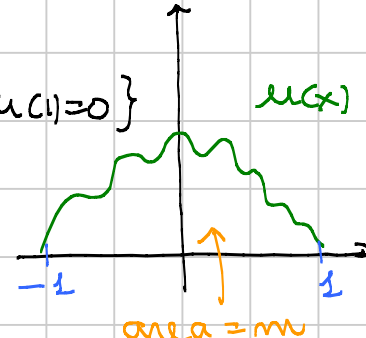


CARTESIAN DIDO'S PROBLEM

Max area with given perimeter or, which is the same, minimizing the perimeter given the area.

Cartesian version

$$\min \left\{ \underbrace{\int_{-1}^1 \sqrt{1+\dot{u}^2} dx}_{\text{length of graph}} : \underbrace{\int_{-1}^1 u dx = m}_{\text{given area}}, u(-1)=u(1)=0 \right\}$$


$$F(u) := \int_{-1}^1 \sqrt{1+\dot{u}^2} dx \quad G(u) = \int_{-1}^1 u dx$$

Bad news: linear growth wrt $\dot{u}$.

Lagrange multipliers: if u is a min point, then

→ either $\delta G(u, v) = 0$ for every v admissible

→ or there exists $\lambda \in \mathbb{R}$ such that $\delta(F + \lambda G) = 0$

First variation of G : $(L_p)' = L_s$
 $0 = 1$ no solution

First variation of $F + \lambda G$: $L(x, s, p) = \sqrt{1+p^2} + \lambda s$

$$(L_p)' = L_s \quad \left(\frac{\dot{u}}{\sqrt{1+\dot{u}^2}} \right)' = \lambda \quad ; \quad \frac{\dot{u}}{\sqrt{1+\dot{u}^2}} = \lambda x + k$$

$$\frac{\ddot{u}^2}{1+\dot{u}^2} = (\lambda x + k)^2; \quad \ddot{u}^2 = (\lambda x + k)^2 + (\lambda x + k)^2 \dot{u}^2$$

$$\dot{u}^2 = \frac{(\lambda x + k)^2}{1 - (\lambda x + k)^2}$$

$$\dot{u} = \pm \frac{\lambda x + k}{\sqrt{1 - (\lambda x + k)^2}}$$

$$= \left(\pm \sqrt{1 - (\lambda x + k)^2} \right)' \frac{1}{\lambda}$$

Finally we obtain that

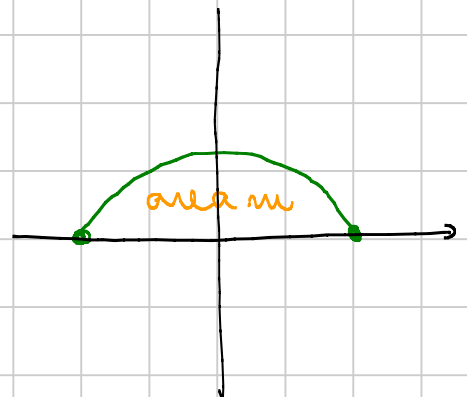
$$\lambda u = \pm \sqrt{1 - (\lambda x + k)^2} + c \leadsto (\lambda u - c)^2 = 1 - (\lambda x + k)^2$$

$$(\lambda x + c)^2 + (\lambda x + k)^2 = 1$$

circle of radius $\frac{1}{\lambda}$ depending on three parameters λ, c, k

we have three equations to fulfil:

$$\begin{cases} u(-1) = 0 \\ u(1) = 0 \\ \int_{-1}^1 u(x) dx = m \end{cases} \rightarrow c = 0 \text{ (center is on y axis)}$$



Not difficult to see that the solution is unique, if it exists.

However: the solution exists if and only if $-\frac{\pi}{2} \leq m \leq \frac{\pi}{2}$

Theorem 1 For every $m \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ the Euler equation admits (at least) one solution $u_0(x)$ given by an arc of a circle with radius $\frac{1}{\lambda}$.

If m is not in the range, then Euler equ. has no solution

Questions : $\rightarrow$ is u_0 a GM?

$\rightarrow$ what happens if $m > \frac{F}{2}$.

sufficient condition

LEMMA (Trivial lemma for constrained minimization)

In a quite general setting, assume that we need to solve

$$\min \{ F(u) : G(u) = m \}$$

Assume that (u_0, λ_0) is a solution to option 2 of Lagrange multipliers, namely u_0 is a stationary point for $F(u) + \lambda_0 G(u)$

Assume that u_0 is a minimum point for $F(u) + \lambda_0 G(u)$.
Then u_0 is a min point for the original problem.

Proof.

$$F(u) = F(u) + \lambda_0 G(u) - \lambda_0 G(u) \geq F(u_0) + \lambda_0 G(u_0) - \lambda_0 G(u)$$

u_0 is $\uparrow$ min point for $F + \lambda G$

$\uparrow$ equal if $G(u_0) = G(u) = m$

This proves that $F(u) \geq F(u_0)$ if $G(u) = m$.

In our case it is enough to check that $u_0(x)$ is a minimum point for

$$(F + \lambda G)u = \int_{-1}^1 \sqrt{1+u^2} + \lambda u$$

[Proof 1] Again by convexity of $L(x, s, p) := \sqrt{1+p^2} + \lambda s$
(the sign of λ does not matter)
wrt the pair (p, s)

Proof 2 Weierstrass excess is ≥ 0 due to convexity in p .
It is enough to find a Weierstrass field

To this end, it is enough to consider vertical translation of $u_0(x)$.

This is true because, if u_0 solves

$$\left(\frac{u}{\sqrt{1+u^2}} \right)' = \lambda$$

then $u_0(x) + c$ is again a solution.

— 0 — 0 —

What happens if $m > \frac{\pi}{2}$?

The min. does not exist (Euler equ. has no solution) and the maximum is represented by a half stadium

The "minimizer" is discontinuous at the boundary.

The price of vertical jump of height J is exactly J



$$\int_0^{\frac{1}{m}} \sqrt{1+u_m^2} = \frac{1}{m} \sqrt{1+m^2} \rightarrow 1$$

$$\int_0^{\frac{1}{m}} \sqrt{1+u_m^2} = \frac{1}{m} \sqrt{1+J^2 m^2} \rightarrow J$$

Sketch of proof Take a level H such that "area above H " is $\frac{\pi}{2}$. Then

- the perimeter above H is at least π
- H can be estimated because

$$2H = \text{area rectangle} \geq m - \frac{\pi}{2}$$

- Perimeter below $H \geq 2H$.

$$\text{Total perimeter} \geq 2H + \pi \geq m - \frac{\pi}{2} + \pi = \text{perimeter of } \frac{1}{2} \text{ stadium}$$

