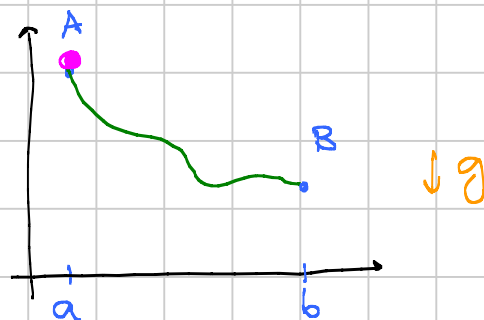


THE BRACHISTOCRONE PROBLEM

Shortest time

- 1- **MODEL** Finding the profile $u(x)$ in such a way that the time from A to B is as short as possible (ball is moving according to gravity)



$(x(t), y(t))$ trajectory $y(t) = u(x(t))$ $\ddot{y}(t) = \ddot{u}(x(t)) \dot{x}(t)$

$$E(t) := \underbrace{\frac{1}{2} (\dot{x}^2 + \dot{y}^2)}_{\text{kinetic}} + gy = E(0) = gA$$

$\uparrow$ speed = 0 at $t=0$

$$\dot{x}^2 + \dot{u}^2 \dot{x}^2 = 2g(A - u(x(t)))$$

$$\dot{x}(t)^2 = 2g \frac{A - u(x(t))}{1 + \dot{u}^2(x(t))}$$

$$\dot{x}(t) = \sqrt{2g} \sqrt{\frac{A - u(x(t))}{1 + \dot{u}^2(x(t))}}$$

$$\int_0^T \sqrt{\frac{1 + \dot{u}^2(x(t))}{A - u(x(t))}} \dot{x}(t) dt = \int_0^T \sqrt{2g} dt = \sqrt{2g} T$$

$y := u(x)$

$$\int_a^b \sqrt{\frac{1 + \dot{u}^2(y)}{A - u(y)}} dy = \sqrt{2g} T$$

Minimizing T is equivalent to the min pbm

$$\min u \left\{ \int_a^b \sqrt{\frac{1 + \dot{u}^2}{A - u}} dx : u(a) = A, u(b) = B \right\}$$

Bad news: 1 - $L(s, p) := \frac{\sqrt{1+p^2}}{\sqrt{A-s}}$ has linear growth

2 - Denominator = 0 when $x = a$ $\leadsto$ improper integrals

We look for a minimizer $u \in C^0([a, b]) \cap C^1((a, b))$

2 - Euler equation Erdmann form $\ddot{u} L_p(u, \dot{u}) = c + L(u, \dot{u})$

$$\frac{\dot{u}^2}{\sqrt{1+\dot{u}^2}} \frac{1}{\sqrt{A-u}} = c + \frac{\sqrt{1+\dot{u}^2}}{\sqrt{A-u}}$$

Brutal move on: $\cancel{\dot{u}^2} = c \sqrt{1+\dot{u}^2} \sqrt{A-u} + 1 + \cancel{\dot{u}^2}$

$$\sqrt{1+\dot{u}^2} \sqrt{A-u} = c \leadsto 1+\dot{u}^2 = \frac{c^2}{A-u}$$

$$\leadsto \dot{u} = \pm \sqrt{\frac{c^2 - (A-u)}{A-u}}$$

ODE solvable by separation of variables

$$\int \sqrt{\frac{A-u}{c^2 - (A-u)}} du = \pm \int dx = \pm x + k$$

Variable change: $A-u = c^2 \sin^2 z \leadsto -du = 2c^2 \sin z \cos z$

$$\pm \int \frac{\cancel{c} \sin z}{\cancel{c} \cos z} 2c^2 \sin z \cos z dz = \pm x + k$$

$$= \pm 2c^2 \int \sin^2 z dz = \pm 2c^2 \left(\frac{1}{2} 2z - \frac{1}{4} \sin(2z) \right)$$

$$= \pm \frac{c^2}{2} (2z - \sin(2z))$$

One more variable change

$$x = \frac{c^2}{2} (\theta - \sin \theta) + k$$

$$u = A - c^2 \sin^2 z = A - \frac{c^2}{2} (1 - \cos \theta)$$

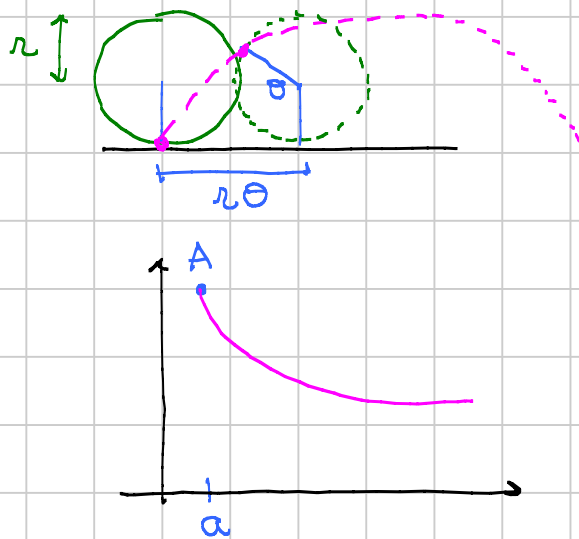
Setting $r := \frac{c^2}{2}$ we found that (choose $k=a$)

$$\begin{cases} x = r(\theta - \sin \theta) + a \\ y = A - r(1 - \cos \theta) \end{cases} \quad \text{Parametrization of a cycloid}$$

Standard form

$$x = r(\theta - \sin \theta) \quad \theta \in [0, 2\pi]$$

$$y = r(1 - \cos \theta)$$



Rank $f(\theta) := \theta - \sin \theta$ is an invertible function (strictly increasing) which admits an inverse $g: \mathbb{R} \rightarrow \mathbb{R}$. From the first eq. we obtain

$$\theta = g\left(\frac{x-a}{r}\right)$$

3-System Does there exist $r > 0$ such that the cycloid passes through (b, B) ?

$$b = r(\theta - \sin \theta) + a$$

$$B = A - r(1 - \cos \theta)$$

$$r(\theta - \sin \theta) = b - a > 0$$

$$r(1 - \cos \theta) = A - B > 0$$

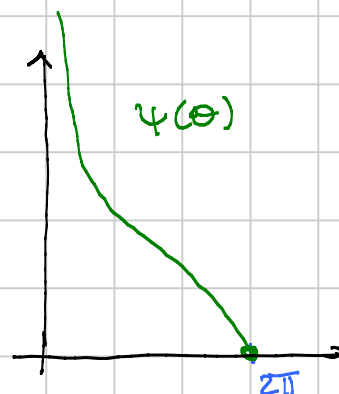
Dividing by r we find

$$\frac{1 - \cos \theta}{\theta - \sin \theta} = \frac{A - B}{b - a} > 0$$

$\psi(\theta)$

Claim: $\psi: (0, 2\pi) \rightarrow (0, +\infty)$ is invertible.

If claim is true, then we can find a unique $\theta \in (0, 2\pi)$ and a unique r that solve the system.



Nice exercise: $\psi'(\theta) < 0 \quad \forall \theta \in (0, 2\pi)$

Theorem 1 Let $b > a$ and let $A > B$. Let $(r_0, \dot{r}_0)$ be the solution to the system.

Let us set

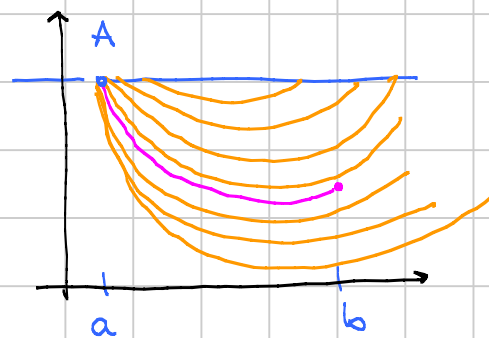
$$u_0(x) := A - r_0 \left(1 - \cos \left(g \left(\frac{x-a}{r_0} \right) \right) \right) \quad \forall x \in [a, b]$$

Then $u_0 \in C^0([a, b]) \cap C^1((a, b))$ is a solution to Euler equ. of the original pbm and satisfies the DBC.

Theorem 2 The function $u_0(x)$ defined above is a GM for the original pbm

Proof 1 Use calibrations $\leadsto$ describing a Weierstrass field (Weierstrass excess is ≥ 0 because $L(x, s, p)$ is convex wrt p)

All the cycloids as $r > 0$ invade the region with $x > a$ and $u < A$



Any other solution in this region satisfies WRF and therefore it is worse than u_0 .

Proof 2 Convexity trick. Let us set $A - u = \frac{1}{2} v^2$ so that

$$-\dot{u} = r\dot{v} \quad \sqrt{\frac{1+\dot{u}^2}{A-u}} = \sqrt{\frac{1+v^2\dot{v}^2}{v^2}} \sqrt{2} = \sqrt{2} \sqrt{\dot{v}^2 + \frac{1}{v^2}}$$

and therefore in the new variable

$$\min \left\{ \int_a^b \sqrt{\frac{1+\dot{u}^2}{A-u}} dx : BC \right\} = \sqrt{2} \left\{ \int_a^b \sqrt{\dot{v}^2 + \frac{1}{v^2}} dx : \text{modified } BC \right\}$$

$v(a) = 0, v(b) = \dots$

and $v_0(x)$ corresponding to $u_0(x)$ is a solution to the new Euler equation.

Surprise $L(s, p) := \sqrt{p^2 + \frac{1}{s^2}}$ is convex wrt (p, s) . (Compute the Hessian matrix)