

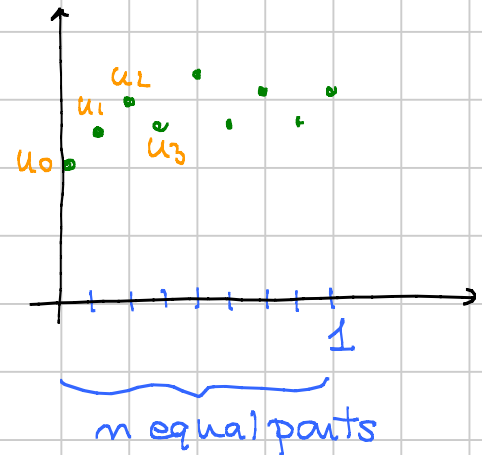
DISCRETE TO CONTINUUM MODELS

Functions $\rightsquigarrow$ values on a grid.

n = points in the grid

Function $\rightsquigarrow (n+1)$ real numbers

$u_0, u_1, \dots, u_n$



$$F(u) = \int_0^1 \dot{u}^2 + u^2$$

$$F_n(u) = \frac{1}{n} \sum_{k=1}^n \left[\left(\frac{u_k - u_{k-1}}{\frac{1}{n}} \right)^2 + u_k^2 \right]$$

$$\int_0^1 u^2 \rightsquigarrow \frac{1}{n} \sum_{k=1}^n u_k^2$$

$$\int_0^1 \dot{u}^2 \rightsquigarrow \frac{1}{n} \sum_{k=1}^n \left(\frac{u_k - u_{k-1}}{\frac{1}{n}} \right)^2$$

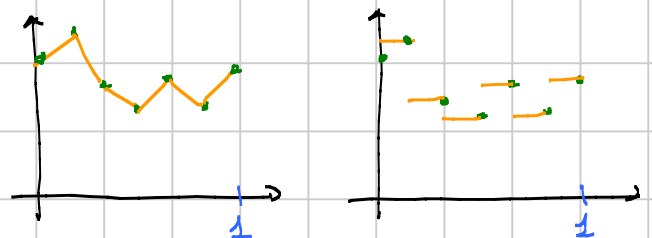
We note that F_n is defined in $\mathbb{R}^{n+1}$, while F is defined in L^2 or C^1 or C^0 or $\mathcal{H}^1$ or...

We expect that $F_n \rightarrow F$ in the sense of Gamma-conv. This requires that F_n and F are defined in the SAME AMBIENT SPACE. We need to identify $\mathbb{R}^{n+1}$ with a suitable subset of L^2 (or C^0).

The are two standard ways to do this

1 - Piecewise affine extension

2 - Piecewise constant



Piecewise affine is good because gives rise to continuous functions and in that case

$$F_n(u) = F(u) \quad \text{as for the "derivative term"}$$

Piecewise constant is good because

$$F_n(u) = F(u) \quad \text{as for the "function term"}$$

and it is always possible, even when u take values in some strange metric space.

Result: $F_n \rightarrow F$ in C^0 or L^2 or ... in the sense of Gamma conv. + equicoerciveness.

Gamma-liminf Take $u_n \rightarrow u_\infty$ in L^2
 $\uparrow$
for every n this is a
piecewise affine functions
on the grid with n samples

Need to prove that $\liminf F_n(u_n) \geq F(u_\infty)$

(Detail: $F_n(u) = +\infty$ when u is not piecewise affine or piecewise constant)

This is easy because $F_n(u_n) = F(u_n)$ if u_n is p.w.a. and F is LSC.

Equicoerciveness The same, because $F_n(u_n) \leq M$

$$\Rightarrow F(u_n) \leq M \Rightarrow \|u_n\|_{L^2} \leq M \quad \|u_n\|_{L^2} \leq M$$

Gamma limsup We need to prove that for every $u \in L^2$ there exists $u_n \rightarrow u$ s.t.

$$\limsup F_n(u_n) \leq F(u)$$

We can assume that $u \in H^1$ and therefore continuous.

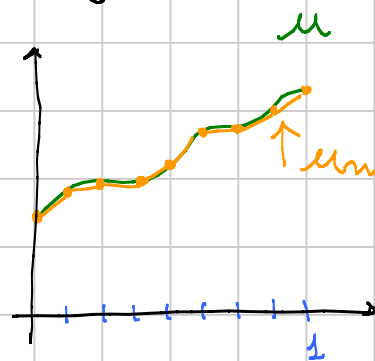
In this case u_n can be defined by connecting the dots !!!

In this case

$$u_n \rightarrow u \text{ unif.}$$

because u is unif. cont.

This guarantees the convergence of the part of $F_n(u_n)$ con to $\int u^2$



As for the other part, it is less that $\int u_n^2$ so again the limsup follows.

— o — o —

Euler equation in the discrete setting.

Notation : set $\varepsilon := \frac{1}{n}$ $D^\varepsilon u(x) = \frac{u(x+\varepsilon) - u(x)}{\varepsilon}$

If u is piecewise constant, then the sum of the derivative part is

$$F_\varepsilon(u) = \int_0^1 |D^\varepsilon u|^2 + u^2$$

Assume that u minimizes $F_\varepsilon(u)$ among all pwc functions

$$\begin{aligned} F_\varepsilon(u+tv) &= \int_0^1 |D^\varepsilon u|^2 + 2t D^\varepsilon u D^\varepsilon v + t^2 |D^\varepsilon v|^2 \\ &\quad + \int_0^1 u^2 + 2t uv + t^2 v^2 \end{aligned}$$

Derivative with $t=0$

$$2 \int_0^1 (D^\varepsilon u D^\varepsilon v + uv) = 0$$

Now we used "discrete integration by parts"

$$\int_0^1 D^\varepsilon f \cdot g = - \int_0^1 f D^{-\varepsilon} g$$

(if for example $f=0$ at the "boundary")

↑
can be easily proved by shifting
indices in summations or
changing variables in integrals

Using this formula we obtain ($g := D^\varepsilon u$, $f := v$)

$$\int_0^1 (D^\varepsilon u D^\varepsilon v + uv) = \int_0^1 -D^{-\varepsilon} D^\varepsilon u \cdot v + uv$$

And therefore

$$D^{-\varepsilon} D^\varepsilon u = u$$

$\rightsquigarrow$

$$\ddot{u} = u$$

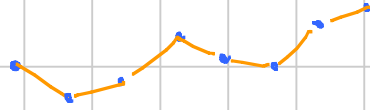
discrete counterpart
of the second derivative

$$D^{-\varepsilon} D^\varepsilon u = \frac{u(x+\varepsilon) + u(x-\varepsilon) - 2u(x)}{\varepsilon^2} \approx \ddot{u}(x)$$

$$\begin{pmatrix} -2 & 1 & & & \\ & 1 & -2 & 1 & 0 \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix}$$

— 0 — 0 —

Second example Approximate the length of a curve by polygons



Given $n+1$ points in the plane or space $P_0, P_1, \dots, P_n$, define

$$F_n(P_0, P_1, \dots, P_n) := \sum_{k=1}^n |P_{k-1} P_k|$$

We would like to say that F converges to the length of a curve functional.

We have to say in which sense $(n+1)$ -uples of points converge to curves.

$(P_0, P_1, \dots, P_n) \rightsquigarrow$ p.w.a. curve defined in $[0, 1]$

So it makes sense to say that a family of such curves converges to some $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ ($\mathbb{R}^3$)

Exercise Prove that $F_n(\gamma) \rightarrow \text{length of } \gamma$ w.r.t to C^0 topology.

