

LINEARIZATION EFFECTS "Small parameters"

Example 1 $\min \left\{ \int_0^1 (\dot{u}^2 + u^4 + \varepsilon u + u^6) dx : u(0)=0, u(1)=0 \right\}$

For $\varepsilon > 0$ fixed existence of minimizers follows from the direct method. Moreover, the min $m_\varepsilon < 0$ because $u \equiv 0$ is a competitor, and it does not solve Euler equ.

Idea: $m_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Problem: find principal part.

Strategy: set $u = \varepsilon v$. The problem becomes

$$\varepsilon^2 \int_0^1 (\dot{v}^2 + \varepsilon^2 \dot{v}^4 + v + \varepsilon^4 v^6) dx$$

$$F_\varepsilon(u) \rightarrow F(u) = \int_0^1 (\dot{v}^2 + v) dx$$

It is reasonable to expect that $\min F_\varepsilon \rightarrow \min F(u)$ with DBC, and this can be easily computed.

This is true if we prove

- ① Γ -convergence
 - ② equicoerciveness of F_ε
- } in $C^0, L^1, L^2, L^{2^*}, \dots$

1.1 Gamma-liminf
then

$F_\varepsilon(u) \geq F(u)$. Therefore if $u_\varepsilon \rightarrow u$

$$\liminf_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0^+} F(u_\varepsilon) \geq F(u)$$

↑
ineq.

↑
F is LSC

①.2 Gamma-Limsup Trivial if $u \in C^1$ (we can take $u_\varepsilon \equiv u$) and C^1 is dense in L^2 wrt the energy $F(u)$.

② We have to prove that $F_\varepsilon(u_\varepsilon) \leq M$ implies existence of a converging subsequence.
Once again it follows from

$$F(u_\varepsilon) \leq F_\varepsilon(u_\varepsilon) \leq M$$

and usual compactness results for $F(u)$.

Example 2 $m_\varepsilon = \min \left\{ \int_0^1 \dot{u}^4 + \varepsilon u + u^6 : u(0) = u(1) = 0 \right\}$

Let us try $u = \varepsilon^\alpha v$

$$\begin{aligned} & \int_0^1 \varepsilon^{4\alpha} \dot{v}^4 + \varepsilon^{1+\alpha} v + \varepsilon^{6\alpha} v^6 \\ & \quad \quad \quad \uparrow \quad \quad \quad \text{choose } 4\alpha = 1+\alpha \Rightarrow \alpha = \frac{1}{3} \\ & = \varepsilon^{\frac{4}{3}} \int_0^1 \dot{v}^4 + v + \varepsilon^{\frac{2}{3}} v^6 \\ & \quad \quad \quad \underbrace{\hspace{10em}}_{F_\varepsilon(v)} \rightarrow F(v) := \int_0^1 \dot{v}^4 + v \end{aligned}$$

As before we have Gamma-convergence and equicoerciveness.

Example 3 $m_\varepsilon := \min \left\{ \int_0^1 (\dot{u}^2 + \dot{u}^8 + u^3 + u^6) dx : u(0) = 0, u(1) = \varepsilon \right\}$

Once again we set $u = \varepsilon v \rightsquigarrow v(0) = 0, v(1) = 1$

$$\begin{aligned} & \int_0^1 (\varepsilon^2 \dot{v}^2 + \varepsilon^8 \dot{v}^8 + \varepsilon^3 v^3 + \varepsilon^6 v^6) dx \\ & \quad \quad \quad \underbrace{\hspace{10em}}_{F_\varepsilon(v)} \rightarrow F(v) := \int_0^1 \dot{v}^2 dx \end{aligned}$$

The minimum of $F(v)$ with BC is $\frac{1}{2}$ and therefore

$$m_\varepsilon \sim \frac{1}{2} \varepsilon^2.$$

Remark. We have also convergence of minimizers

$$v_\varepsilon(x) \rightarrow x \text{ uniformly.}$$

Again: one has to prove Gamma-conv. and equicoerciveness.

Example 4 $m_\varepsilon := \min \left\{ \int_0^1 \sinh(u^2) + \sin(u^2) : u(0) = u(1) = \varepsilon \right\}$

As before $u = \varepsilon v$

$$\begin{aligned} \int_0^1 \sinh(\varepsilon^2 \dot{v}^2) + \sin(\varepsilon^2 v^2) &= \\ &= \varepsilon^2 \int_0^1 \frac{\sinh(\varepsilon^2 \dot{v}^2)}{\varepsilon^2} + \frac{\sin(\varepsilon^2 v^2)}{\varepsilon^2} \\ &\quad \underbrace{\hspace{10em}}_{F_\varepsilon(v) \rightarrow F(v) := \int_0^1 \dot{v}^2 + v^2} \end{aligned}$$

Again we can prove Gamma-conv. and equicoerciveness.

$$F_\varepsilon(v) \geq \int_0^1 (\dot{v}^2 - 1) dx$$

$\uparrow$
 $\sinh x \geq x \text{ when } x \geq 0$

$$\frac{\sinh x}{x} \geq -1 \text{ when } x \geq 0$$

This provides equicoerciveness, and the key step in the liminf inequality.

Limsup inequality is easy in C^1 , which is dense in energy.

Example 5

$$u_\varepsilon := \left\{ \int_0^1 \sin(u^2) + \sin R(u^2) : u(0) = u(1) = \varepsilon \right\}$$

In this case the min. does not exist for every $\varepsilon > 0$ and the infimum is actually always -1 , because it is the min. of the relaxation which is

$$\int_0^1 -1 + \sin R(u^2)$$

Example 6

$$\int_0^1 \dot{u}^2 + u^4 \quad u(0) = u(1) = \varepsilon$$

$$u = \varepsilon v \quad \rightsquigarrow \quad \int_0^1 \varepsilon^2 \dot{v}^2 + \varepsilon^4 v^4 \quad v(0) = v(1) = 1$$

$$= \varepsilon^2 \underbrace{\int_0^1 \dot{v}^2 + v^4}_{F_\varepsilon(v)}$$

$$F_\varepsilon(v) \rightarrow F(v) = \int_0^1 \dot{v}^2$$

↑
has min = 0.

$$\int_0^1 \varepsilon^2 \dot{v}^2 + \varepsilon^4 v^4 = \varepsilon^4 \underbrace{\int_0^1 \frac{\dot{v}^2}{\varepsilon^2} + v^4}_{F_\varepsilon(v)}$$

$$F_\varepsilon(v) \rightarrow F(v)$$

where

$$F(v) := \begin{cases} \int_0^1 v^4 & \text{if } v \text{ is constant} \\ +\infty & \text{otherwise (in } L^2 \text{ or } C^0 \text{ or...)} \end{cases}$$

Easy : • limsup inequality : can take $u_\varepsilon := u$
• equicoerciveness

$$F_\varepsilon(v_\varepsilon) \leq M \rightsquigarrow \|\dot{v}_\varepsilon\|_{L^2} \leq \varepsilon^2 M$$

• liminf ineq : follows from previous estimate
(in the limit $\dot{v}_\varepsilon \rightarrow 0$ strongly, so that the limit is constant)