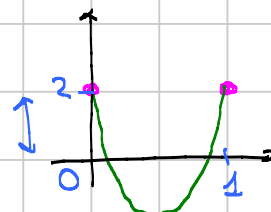


OBSTACLE PROBLEMS

Motivating example $F(u) = \int_0^1 (\dot{u}^2 + 100u) dx$

with DBC $u(0) = 2, u(1) = 2.$



Obstacle: $u(x) \geq 0 \quad \forall x \in [0, 1]$
or $u(x) \geq \sin(12x)$

Obstacles on the function: conditions such as

$$u(x) \geq \varphi(x)$$

$$\forall x \in [a, b]$$

$$\varphi_1(x) \leq u(x) \leq \varphi_2(x)$$

$$\forall x \in [a, b]$$

$$u(x) \geq \varphi(x)$$

$$\forall x \in [c, d] \subseteq [a, b]$$

↑
given

Obstacles on the derivative: $u'(x) \geq -1 \quad \forall x \in [a, b]$
 $u'(x) \geq \varphi(x) \quad \forall x \in [c, d] \subseteq [a, b].$
—○—○—

DIRECT METHODS

• Obstacles on the function pass to the limit w/ unif. convergence $\leadsto$ NO PBM.

$$\min \{ F(u) : \text{DBC} + u(x) \geq \sin(10x) \text{ in } [0, 1] \}$$

Take a min. sequence $u_n \leadsto$ obtain a bound on $\|u_n\|_{L^2}$

$\leadsto$ deduce a bound on $\|u_n\|_{\infty}$ because of DBC

$\leadsto$ apply AA. and up to subsequences $u_n \rightarrow u_{\infty}$ unif
 $\dot{u}_n \rightarrow \dot{v}_{\infty}$

$\leadsto$ deduce that $u_{\infty}(x) \geq \sin(10x)$ in $[0, 1]$

$\Rightarrow$ EXISTENCE 😊

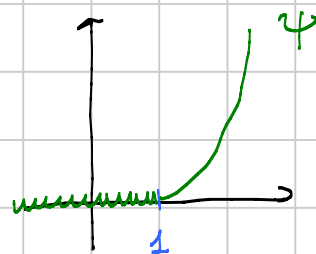
- Obstacles on the derivative are more difficult to deal with
BUT
we can use an integral trick.

Assume $v_m \rightarrow v_\infty$ and $v_m(x) \leq 1$ in $[a, b]$

Can we deduce that $v_\infty(x) \leq 1$ in $[a, b]$? YES

Proof. Take $\psi: \mathbb{R} \rightarrow \mathbb{R}$ as below

- convex
- $= 0$ before 1
- > 0 after 1



For every $m \in \mathbb{N}$ we have that $\int_a^b \psi(v_m) dx = 0$.

Since ψ is convex

$$0 \leq \int_a^b \psi(v_\infty) dx \leq \liminf_{m \rightarrow \infty} \int_a^b \psi(v_m) dx = 0$$

Therefore $\psi(v_\infty) = 0$ almost everywhere, and this means that $v_\infty \leq 1$ almost everywhere.

— 0 — 0 —

If the obstacle is $u_i(x) \geq \sin x$ a similar procedure works starting from

$$\int_a^b \psi(u_m(x) - \sin x) dx$$

— 0 — 0 —



Next steps:

- Euler equ.
- regularity

Assume obstacle on the function $u(x) \geq \varphi(x)$

Assume that u is a minimizer for $F(u) = \int_a^b L(x, u, \dot{u}) dx$

Consider $\varphi(t) := F(u + tv)$ with $v(x) \geq 0$ in $[a, b]$ and $t \geq 0$

By minimality we deduce that $\varphi'(0) \geq 0$



$$\leadsto \int_a^b (L_s(x, u, \dot{u})v + L_p(x, u, \dot{u})\ddot{v}) \geq 0$$

$$\leadsto \int_a^b (-[L_p(x, u, \dot{u})]' + L_s(x, u, \dot{u}))v \geq 0 \quad \forall v \geq 0$$

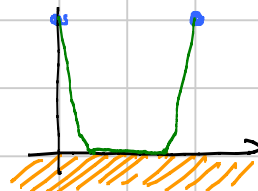
Variant of FLCV $\Rightarrow [L_p(x, u, \dot{u})]' \leq L_s(x, u, \dot{u})$

with equality in all points where $u(x) > \varphi(x)$ (in that case we can use any v in a neigh of x).

In conclusion

- $\rightarrow$ inequality everywhere
- $\rightarrow$ equality outside the contact region

Back to the example



$$L(x, s, p) = p^2 + 100s$$

$$(2\ddot{u})' \leq 100$$

$\uparrow$
at least in weak sense
that $\ddot{u} \in H^1$

Contact condition

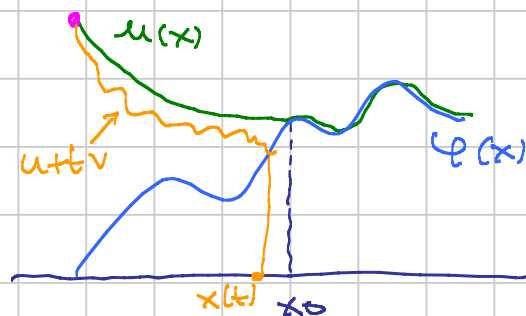
If $L(x, s, p)$ is strictly convex w.r.t p and the obstacle $u(x) \geq \varphi(x)$ is of class C^2 , then in any contact point x_0 it turns out that

$$\ddot{u}(x_0) = \ddot{\varphi}(x_0)$$

"Proof" Assume a simple situation where

$$\begin{aligned} u(x) &> \varphi(x) & \text{for } x < x_0 \\ u(x) &= \varphi(x) & \text{for } x > x_0 \end{aligned}$$

We claim that



$$L_p(x_0, u(x_0), \dot{u}(x_0)) (\dot{u}(x_0) - \dot{\varphi}(x_0)) = L(x_0, u(x_0), \dot{u}(x_0)) - L(x_0, u(x_0), \dot{\varphi}(x_0))$$

If L is strictly convex, this can be true $\Leftrightarrow \dot{u}(x_0) = \dot{\varphi}(x_0)$

As in the theory of point-to-curve plan we consider a variation $v(x)$, with any sign, we consider $x(t)$ in such a way that

$$u(x(t)) + t v(x(t)) = \varphi(x(t))$$

We compare $F(u)$ with $F(u+tv)$

$$\varphi(t) := \int_a^{x(t)} L(x, u+tv, \dot{u}+t\dot{v}) + \int_{x(t)}^b L(x, \varphi, \dot{\varphi})$$

Standard calculation

$$\begin{aligned} \varphi'(0) &= L(x_0, u(x_0), \dot{u}(x_0)) \dot{x}(0) + \int_a^{x_0} L_s(\dots) v + L_p(\dots) \dot{v} - \\ &\quad - L(x_0, \varphi(x_0), \dot{\varphi}(x_0)) \dot{x}(0) = 0 \end{aligned}$$

$$\dot{x}(0) [L(x_0, u(x_0), \dot{u}(x_0)) - L(x_0, u(x_0), \dot{\varphi}(x_0))] = - \int_a^{x_0} (L_s v + L_p \dot{v})$$

$$= - \int_a^{x_0} (L_s - \underbrace{[L_p]'}_0) v - \underbrace{[L_p v]_a}_{\substack{\uparrow \\ \text{when } x=a \text{ something vanishes}}}^{x_0}$$

$$= L_p(x_0, u(x_0), \dot{u}(x_0)) v(x_0)$$

It remains to compute $\dot{x}(0)$:

$$[u(x(t)) + t v(x(t))] = [\varphi(x(t))]'$$

$$\dot{u}(x(t)) \dot{x}(t) + v(x(t)) + t \dot{v}(x(t)) \dot{x}(t) = \dot{\varphi}(x(t)) \dot{x}(t)$$

Set $t=0$: $\dot{u}(x_0) \dot{x}(0) + v(x_0) = \dot{\varphi}(x_0) \dot{x}(0)$

$$\leadsto \dot{x}(0) = - \frac{v(x_0)}{\dot{u}(x_0) - \dot{\varphi}(x_0)} \quad \uparrow \text{ if this is } \neq 0.$$

In conclusion

$$- \frac{\cancel{v(x_0)}}{\dot{u}(x_0) - \dot{\varphi}(x_0)} [L(x_0, u, \dot{u}) - L(x_0, \varphi, \dot{\varphi})] = L_p(x_0, u, \dot{u}) \cancel{v(x_0)}$$

Rearranging the terms we obtain the equality.

Remark To be rigorous, we need to prove the existence of $x(t)$ by means of the implicit function theorem, as in the point-to-curve problem. Lecture

$$\Phi(t, x) := u(x) + t v(x) - \varphi(x) = 0$$

Regularity of minimizer with obstacle $u(x) \geq \varphi(x)$

- If $\varphi \in C^1$, then $u \in C^1$
- If $\varphi \in C^{1,1}$ (φ is Lipschitz), then $u \in C^{1,1}$ (if L_φ is strictly increasing)
- If $\varphi \in C^\infty$, then $u \in C^{1,1}$

In the motivating example, the solution is "only" $C^{1,1}$ and $\varphi(x) \in C^\infty$.

$\ddot{u}$ is discontinuous.

