

GEODESICS Finding the shortest curve between 2 given points.

PLANE (or $\mathbb{R}^n$) $\gamma(t) = (u(t), v(t))$ $\gamma: [a, b] \rightarrow \mathbb{R}^2$

$$\min \left\{ \int_a^b \sqrt{\dot{u}^2 + \dot{v}^2} dt : \gamma(a) = A, \gamma(b) = B \right\}$$

$\in \mathbb{R}^2$ $\in \mathbb{R}^2$

$$L(p_1, p_2) := \sqrt{p_1^2 + p_2^2}$$

Bad news: linear growth

Euler equ: $\left(\frac{\dot{u}}{\sqrt{\dot{u}^2 + \dot{v}^2}} \right)' = 0$ $\left(\frac{\dot{v}}{\sqrt{\dot{u}^2 + \dot{v}^2}} \right)' = 0$

$$\frac{\dot{u}}{\sqrt{\dot{u}^2 + \dot{v}^2}} = c_1$$

$$\frac{\dot{v}}{\sqrt{\dot{u}^2 + \dot{v}^2}} = c_2$$

After some discussion (the denom. could be $= 0$) it is reasonable to expect that

$$\ddot{u}(t) = c \dot{v}(t) \quad \forall t \in [a, b]$$

from which we deduce that $\gamma(t)$ is a parametrization of the segment AB.

Remark Any injective param. of AB is good.

This gives NON UNIQUENESS



Two calibrations by null Lagrangians

Calibration 1 Let $B-A = (\alpha, \beta)$. Then

$$\alpha p_1 + \beta p_2 \stackrel{\substack{\uparrow \\ \text{c.s.}}}{\leq} \sqrt{\alpha^2 + \beta^2} \sqrt{p_1^2 + p_2^2}$$

Therefore

$$\begin{aligned} \int_a^b \sqrt{\dot{u}^2 + \dot{v}^2} dt &\geq \frac{1}{\sqrt{\alpha^2 + \beta^2}} \int_a^b \alpha \dot{u} + \beta \dot{v} \\ &= \frac{1}{AB} \left(\alpha [\underbrace{u(t)]_a^b}_{\alpha} + \beta [\underbrace{v(t)]_a^b}_{\beta} \right) = AB \end{aligned}$$

This proves that the segment minimizes the length.

We have equality iff $(\alpha, \beta) = \lambda (\dot{u}, \dot{v}) \leadsto$ Euler equation

Calibration 2 Use that

$$u\dot{u} + v\dot{v} \stackrel{\substack{\uparrow \\ \text{c.s.}}}{\leq} \sqrt{u^2 + v^2} \cdot \sqrt{\dot{u}^2 + \dot{v}^2}$$

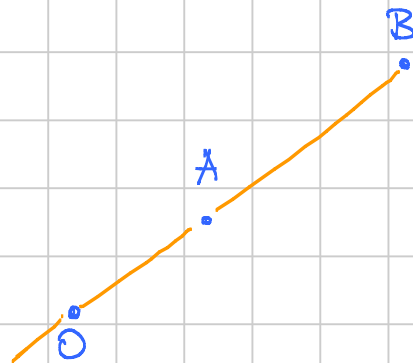
Therefore

$$\begin{aligned} \int_a^b \sqrt{\dot{u}^2 + \dot{v}^2} dt &\geq \int_a^b \left(\frac{u\dot{u}}{\sqrt{u^2 + v^2}} + \frac{v\dot{v}}{\sqrt{u^2 + v^2}} \right) dt \\ &= \int_a^b \frac{d}{dt} \sqrt{u^2 + v^2} dt = \left[\sqrt{u^2 + v^2} \right]_a^b \end{aligned}$$

Assume that A, B and the origin lie on the same line

In this case

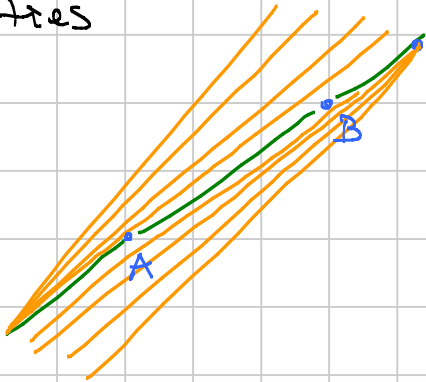
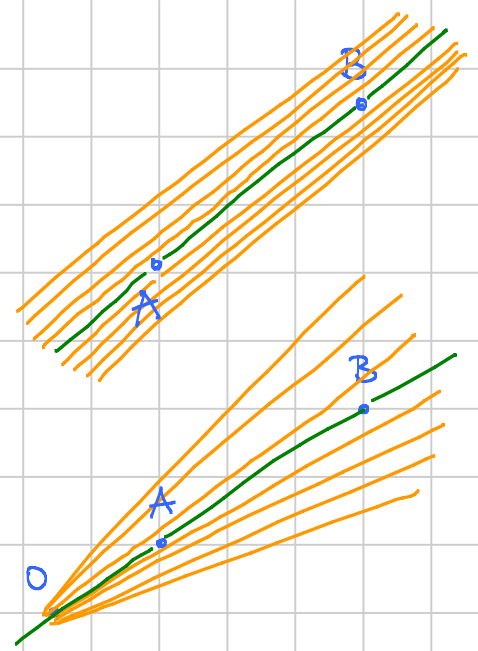
$$\left[\sqrt{u^2 + v^2} \right]_a^b = OB - OA = AB$$



From the geometrical point of view

- Calibration 1 comes from a family of parallel lines
- Calibration 2 comes from the lines originating from 0

Of course there are many other possibilities



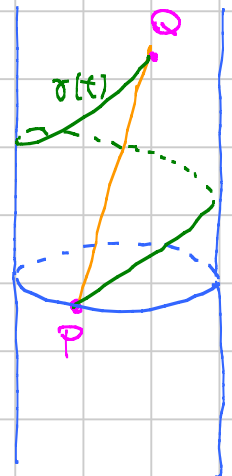
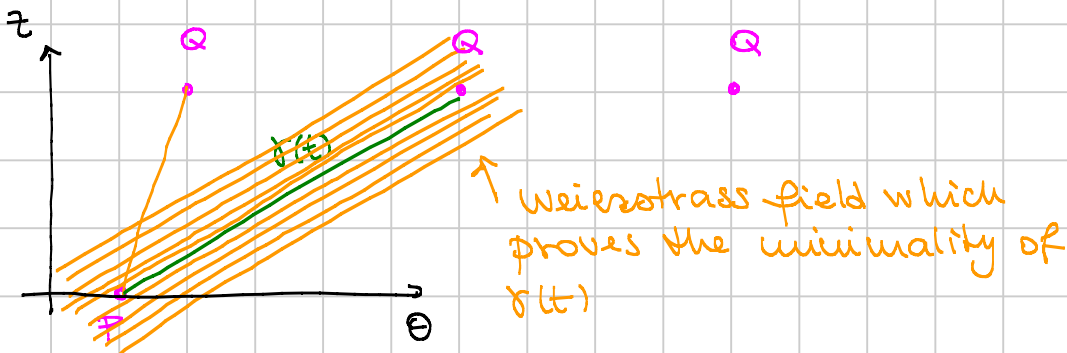
CYLINDER In $\mathbb{R}^3$ $x^2 + y^2 = 1$,

A curve with values in the cylinder is

$$\gamma(t) = (\cos \theta(t), \sin \theta(t), z(t)) \quad t \in [0, 1]$$

and
$$F(\gamma) = \int_0^1 \sqrt{\dot{\theta}^2 + \dot{z}^2} dt$$

Situation is similar to the plane, but



$\gamma(t)$ is a SLM but not a GM. It is the GM among all curves turning once around the cylinder

SPHERE

Let us use spherical coordinates

$$x = \rho \cos \varphi \cos \theta$$

$$\theta \in [0, 2\pi]$$

$$y = \rho \cos \varphi \sin \theta$$

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

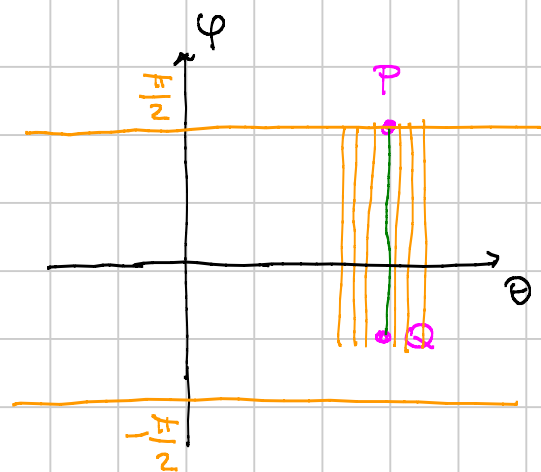
$$z = \rho \sin \varphi$$

sphere $\Rightarrow \rho = \text{constant} = 1$ Curve $\leadsto \theta(t), \varphi(t)$

$$\text{length} := \int_a^b \sqrt{\dot{\varphi}^2 + \cos^2 \varphi \dot{\theta}^2} dt$$

We can assume wlog that $P = \text{North Pole}$

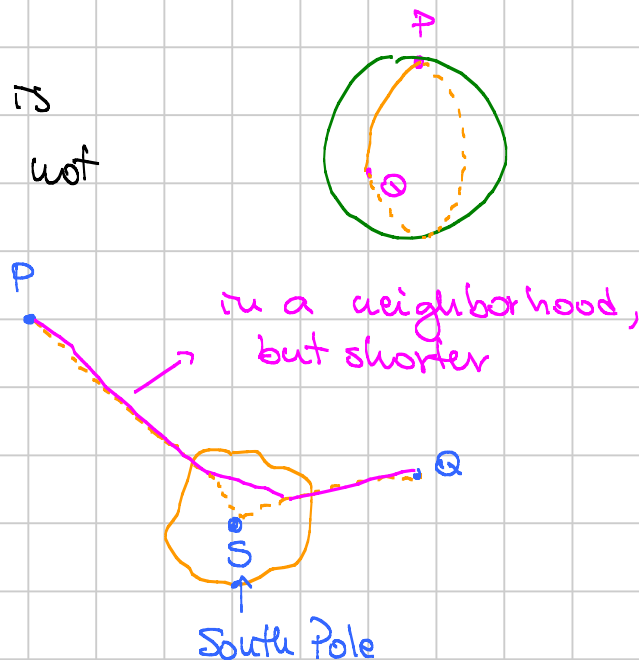
$$\begin{aligned} \int_a^b \sqrt{\dot{\varphi}^2 + \cos^2 \varphi \dot{\theta}^2} dt &\geq \int_a^b \sqrt{\dot{\varphi}^2} \\ &= \int_a^b |\dot{\varphi}| \\ &\geq \left| \int_a^b \dot{\varphi}(t) dt \right| \\ &= |\varphi(P) - \varphi(Q)| \end{aligned}$$



The geodesic is the meridian (the shortest one).

What about the other meridian? It is a solution to Euler eqn, but it is not a WLM.

In some sense, the South Pole is the conjugate point of the North Pole.



MIRACLE OF GEODESICS

Let us consider geodesics on any surface

$$\gamma: [a, b] \rightarrow \mathbb{R}^n \quad \text{with} \quad \gamma(t) \in S \quad \forall t \in [a, b]$$

$\uparrow$
given

$$F_1(\gamma) := \int_a^b \|\dot{\gamma}(t)\| dt$$

$$F_2(\gamma) := \int_a^b \|\dot{\gamma}(t)\|^2 dt$$

$\uparrow$
quadratic growth
 $\leadsto$ very good

Miracle The (unique) minimizer of F_2 is also one of the minimizers of $F_1(\gamma)$.

$$F_1(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \cdot 1 dt \stackrel{\text{C.S.}}{\leq} \left(\int_a^b \|\dot{\gamma}(t)\|^2 dt \right)^{1/2} \cdot \sqrt{b-a}$$

This proves that

$$\min F_1 \leq \sqrt{b-a} \left(\min F_2 \right)^{1/2}$$

But we have equality if we assume the following.

Lemma (reparametrization Lemma) Given any curve $\gamma: [a, b] \rightarrow \mathbb{R}^n$ there exists a nondecreasing function $\varphi: [a, b] \rightarrow [a, b]$ with $\varphi(a) = a$ and $\varphi(b) = b$ such that $\gamma_1(t) := \gamma(\varphi(t))$ has constant speed, namely $\|\dot{\gamma}_1(t)\| = \text{constant}$ for almost every $t \in [a, b]$.

Remark If $\gamma \in C^1$, then we cannot conclude that $\gamma \in C^1$!!

$\nearrow$ is the image of a C^1 curve, but not the image of C^1 curve with constant speed.