

RELAXATION OF NONCONVEX FUNCTIONALS

Def. (Convex envelop of a function)

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be any function. We call convexification (or convex envelope) of ψ the function

$$\psi^{**}(x) = \sup \{ \varphi(x) : \varphi \text{ is a convex function } : \mathbb{R} \rightarrow \mathbb{R} \text{ with } \varphi(x) \leq \psi(x) \forall x \in \mathbb{R} \}$$

Remark Either we allow function φ with values in $\mathbb{R} \cup \{-\infty\}$ or we need to assume the existence of at least one convex $\varphi \leq \psi$.

Prop. (Properties of ψ^{**})

Assume $\exists \varphi \leq \psi$ with φ convex.

Then

(i) ψ^{**} is a convex function

(general fact: the supremum of any family of convex functions is convex because $\text{epigraph} = \bigcap \text{epigraphs}$)

(ii) $\psi^{**}(x) \leq \psi(x)$ for every $x \in \mathbb{R}$,

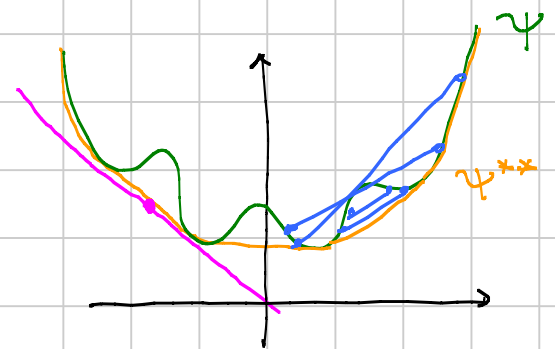
(iii) the sup in the definition is a maximum

(iv) equivalent definition

$$\psi^{**}(x) := \sup \{ ux + n : uy + n \leq \psi(y) \text{ for every } y \in \mathbb{R} \}$$

(v) equivalent definition:

$$\psi^{**}(x) := \sup \{ ux + n : \text{line } uy + n \text{ intersect graph of } \psi \text{ both before and after } x \}$$



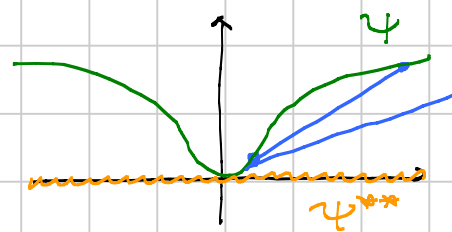
Remark ① Definition (v) is equivalent just in dim 1

② Let us consider

$$\psi(p) := \arctan(p^2)$$

$$\psi(p) := \log(1+p^2)$$

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Theorem (Relaxation in the general case)

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be any function s.t.

(i) ψ is continuous (maybe not needed)

(ii) ψ has quadratic growth (again superlinear is enough)

Then the relaxation of

$$F(u) := \int_a^b \psi(u) dx$$

$$u \in C^1 \text{ or } C^\infty$$

(and $+\infty$ otherwise)

is

$$G(u) := \begin{cases} \int_a^b \psi^{**}(u) dx \\ +\infty \end{cases}$$

$$\text{if } u \in H^1$$

(even in this case it might be $+\infty$)

otherwise

Proof Again step 1, 2 and 3 as before

Step ① Trivial because $\psi \geq \psi^{**}$

Step ② Same as before. just need to remark that ψ^{**} has quadratic growth ($\psi(p) \geq Ap^2 - B$)

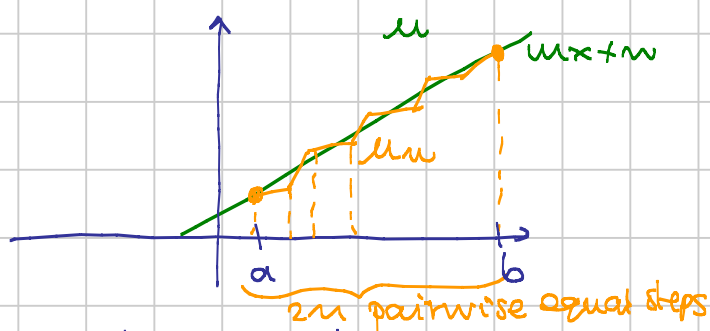
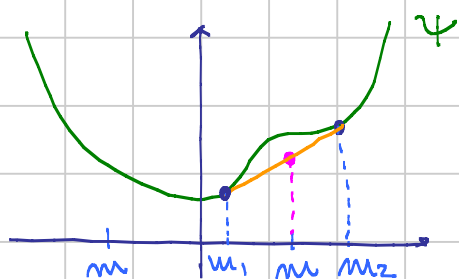
↑ convex and therefore
competes in defining ψ^{**})

Step ③.1 As before, the set of p.w. affine functions is dense in energy wrt G .

(we proved it exploiting the convexity of ψ^{**})

Step (3.2) We need a recovery sequence for every u p.w. affine.

Let us assume first that u is affine: $u(x) = mx + n$



If $\psi(u) = \psi^{**}(u)$, then there is nothing to do: $u_n \equiv u$ does the job.

So wlog $\psi^{**}(u) < \psi(u)$. Assume for a while that there exist m_1 and m_2 such that

$(u, \psi^{**}(u))$ lies on the segment $(m_1, \psi(m_1)), (m_2, \psi(m_2))$

This means that

$$u = \lambda m_1 + (1-\lambda)m_2$$

$$\psi^{**}(u) = \lambda \psi(m_1) + (1-\lambda)\psi(m_2)$$

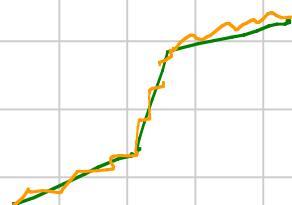
Then we can approximate u by means of a sequence of p.w. affine functions with slopes $\in \{m_1, m_2\}$

$$F(u_n) = \lambda(b-a)\psi(m_1) + (1-\lambda)(b-a)\psi(m_2)$$

$$= (b-a)\psi^{**}(u) = \int_a^b \psi^{**}(u) = G(u)$$

If $\psi^{**}(u)$ is not on the segment, then it is in some segment up to a small ε (because of property (iv) of ψ^{**}).

If u is just p.w. affine, then we do the same procedure in each subinterval.

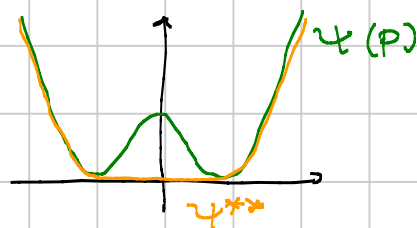


Example

$$\inf \left\{ \underbrace{\int_0^{10} (1-u^2)^2 + (u - \cos x)^4 dx}_{F(u)} : u \in C^1([0,1]) \right\}$$

$$\psi(p) := (1-p^2)^2$$

ψ has quadratic growth



$\Rightarrow F(u)$ is coercive ($F(u_n) \leq M \Rightarrow u_p$ to subseq.

$$u_n \rightarrow u_\infty \text{ unif.}$$

$$u_n \rightarrow u_\infty \text{ weakly in } L^2)$$

The relaxation of $F(u)$ in C^0 or L^{24} is

$$G(u) := \int_0^{10} \psi^{**}(u) + (u - \cos x)^4 dx \quad +\infty \text{ when there are jumps.}$$

(because the second integral is continuous)

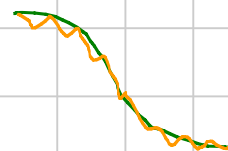
Since F is coercive, then $\inf F = \min G$ because

$u(x) \equiv \cos x$ is a min. point (actually the unique min point)

The point is that $|u'(x)| \leq 1$ and hence $\psi^{**}(u'(x)) \equiv 0$.

Of course it is not a min. for $F(u)$. On the other hand, if u_n is any min. seq. for $F(u)$ (meaning that $F(u_n) \rightarrow \inf$), then

$$u_n(x) \rightarrow \cos x \text{ uniformly}$$



It could be interesting to understand the limit of $u_n(x)$

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