

RELAXATION OF INTEGRAL FUNCTIONALS

Let us consider $F(u) := \int_a^b \psi(u') dx$ when $u \in C^1$ or $u \in C^\infty$.

Goal: extending $F(u)$ to L^2, L^1, C^0 through relaxation.

Key words : $\rightarrow$ convexity
 $\rightarrow$ growth assumptions

Theorem (convex case) Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with

$\rightarrow \psi$ is convex

$\rightarrow \psi$ has quadratic growth at $\pm\infty$ (superlinear growth is enough)

Then the relaxation of $F(u)$ to any L^p or C^0 is

$$F(u) := \int_a^b \psi(u') dx \quad \text{if } u \text{ has a weak derivative } u'$$

$\uparrow$ even this case could give $+\infty$

$$F(u) := +\infty \text{ otherwise}$$

Recall a strategy for proving that some $G(u)$ is the relaxation of $F(u)$

① $F(u) \geq G(u)$ for every u admissible

② $G(u)$ is LSC

③ There exists D dense in energy with respect to G such that for every $u \in D$ there exists $u_n \rightarrow u$ (in the ambient space) and $F(u_n) \rightarrow G(u)$
 (but $\limsup_{n \rightarrow \infty} F(u_n) \leq G(u)$ is enough)

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Proof Let $G(u)$ denote what we stated to be the relaxation.

Step ① is trivial because $F(u) = +\infty$ outside C^+ or C^∞ .

Step ② is more or less the LSC result of previous lectures.

To be more precise, let $u_n \rightarrow u_\infty$ be any sequence.

We need to prove that

$$\liminf_{n \rightarrow \infty} G(u_n) \geq G(u_\infty)$$

We can assume wlog that $\liminf = \lim < +\infty$. By the growth assumptions this implies that

$$\|u_n\|_{L^2} \leq M \quad \left(\psi(p) \geq \underset{>0}{A} p^2 - B \right)$$

So up to subsequences $u_n \rightarrow u_\infty$ in $L^2(a,b)$.

Now we can apply the LSC result (thanks to convexity) and deduce that

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(u_n) dx \geq \int_a^b \psi(u_\infty) dx$$

(just requires $u_n \rightarrow u_\infty$ in L^1)

Step ③.1 A good subset dense in energy wrt G is the set of piecewise affine functions.

For every $u \in L^p, L^1, C^0$ we need $u_n \rightarrow u$

such that $G(u_n) \rightarrow G(u)$ $\uparrow$
piecewise affine

Again $\limsup G(u_n) \leq G(u)$ is enough

Strategy: CONNECT THE DOTS !!

wlog $u \in C^0$ (if not, $G(u) = +\infty$
because $G(u) < +\infty \Rightarrow u \in H^1$
because of quadratic growth)

Simple exercise: $u_n \rightarrow u$ unif.

Moreover: $G(u_n) \leq G(u) \rightsquigarrow \limsup$ ineq.

Why $G(u_n) \leq G(u)$? Two possible arguments

→ this is true on each subinterval

$$\int_{x_i}^{x_{i+1}} \underbrace{\psi(\text{line})}_{\frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}} dx \leq \int_{x_i}^{x_{i+1}} \psi(u) dx$$

(The line is the best way to fit BCs in the subinterval)

→ Jensen inequality on each interval (Jensen inequality for integrals)

Step 3.2 Every u which is piecewise affine can be approximated in L^1, L^p, C^0 by a sequence $u_n \rightarrow u$ with $u_n \in C^1$ or C^∞ in such a way that

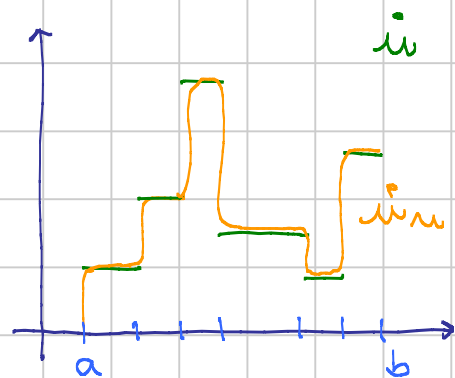
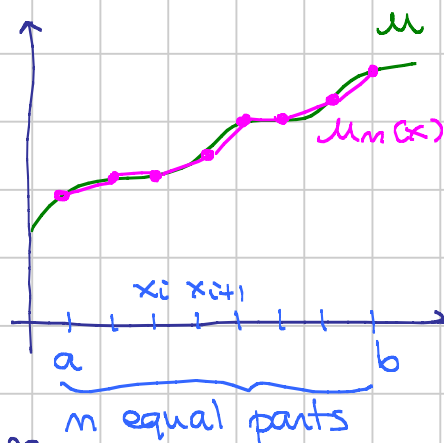
$$\mathbb{F}(u_n) \rightarrow G(u)$$

We can approximate $\tilde{u}_n \rightarrow u$
 $\in C^0 \text{ or } C^\infty$ in any L^p space

Define

$$u_n(x) := u(a) + \int_a^x \tilde{u}_n(t) dt$$

and the proof is complete. $\square$



Remark 1 Things are very different if ψ has just linear growth.

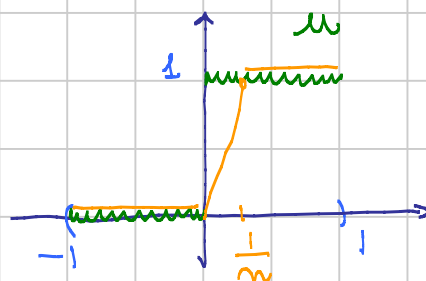
Classical example $\psi(p) := \sqrt{1+p^2}$



In this case there are discontinuous functions u for which $\bar{F}(u) < +\infty$

Can approximate u in any L^p with $p < +\infty$ by a sequence $u_n \in C^1$ with

$$\int_{-1}^1 \psi(u_n) dx \leq M$$



Idea: take u_n as above

$$\int_{-1}^1 \psi(u_n) dx = \int_{-1}^0 1 dx + \int_0^1 1 dx + \int_0^{1/n} \sqrt{1+n^2} dx$$

$\underbrace{\quad}_{=1} \quad \underbrace{\quad}_{\leq 1} \quad \underbrace{\int_0^{1/n} \sqrt{1+n^2} dx}_{\substack{\frac{1}{n} \sqrt{1+n^2} \\ \text{bounded}}}$

Future exausion In this case the relaxation is finite in the BV space
 $\uparrow$
 bounded variation



Remark 2 Let us consider $\psi(p) = e^{-p}$
 Problem: Lack of growth as $p \rightarrow +\infty$.

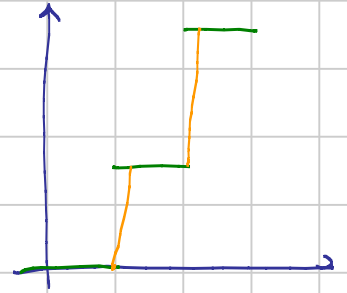
In this case $\bar{F}(u) = \uparrow$ for every nondecreasing function
 the same

Let us consider u as before and $u_n \rightarrow u$ as before

$$\int_{-1}^1 \psi(u_n) dx = \int_{-1}^0 + \int_0^1 + \underbrace{\frac{1}{n} e^{-n}}_0 \leq 2$$

The same is true for any function with two jumps in ascending order.

Therefore any increasing piecewise constant function pays less than 2.



Conclusion: increasing pwc functions are dense in energy in the set of nondecreasing functions.

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