

$$F(u) = \int_a^b L(x, u, \dot{u}) dx$$

Philosophy: $\rightarrow$ growth assumption on L wrt p , for example

$$L(x, s, p) \geq \psi(p) \quad \forall (x, s, p) \in [a, b] \times \mathbb{R}^2$$

with $\psi: \mathbb{R} \rightarrow \mathbb{R}$ with superlinear growth

$\Rightarrow$ weak compactness in $L^1((a, b))$ of $\dot{u}_n$, where u_n is any sequence with $F(u_n) \leq M$.

$\rightarrow$ As before + some (integral or pointwise) control on $u_n \Rightarrow$ (up to subsequences) $u_n \rightarrow u_\infty$ uniformly.

$\rightarrow$ Same sort of convexity of L wrt p , for example

$$L(x, s, p) := \underbrace{\psi(p)}_{\text{convex}} + g(x, s)$$

$\Rightarrow$ LSC.

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Back to motivational example

$$F(u) = \int_0^1 \dot{u}^4 + u^8 \quad \text{with DBC } u(0) = u(1) = 2016$$

Direct method $\Rightarrow$ existence of GM.

Derive Euler equation $\varphi(t) = F(u + tv) = \int_0^1 (\dot{u} + t\dot{v})^4 + (u + tv)^8$

and compute

$$\varphi'(0) = 4 \int_0^1 \dot{u}^3 \dot{v} + 8 \int_0^1 u^7 v = \delta F(u, v)$$

Remark This requires switching integral and derivative, which in turn requires some unif. bounds

$$\varphi'(t) = \int_0^1 \underbrace{(u+tv)^3 \tilde{v} + 8(u+tv)^2 v}_{\text{we need a domination of this indep. of } t. \text{ This is possible because we can control } u^3 \text{ in terms of } u^4.}$$

After integration by parts (or the definition of weak derivative) we end up with

$$(4u^3)' = 8u^2$$

$$(u^3)' = 2u^2$$

Formally this means that

$$\underbrace{3u^2 u'}_{L^p} = 2u^2$$

Question : what about the regularity of u ? Can we say that $u \in C^\infty$? Or even $u \in C^2$?

We know that $u \in L^4 \Rightarrow u \in C^0$ (and actually $\frac{3}{4}$ -Hölder cont.)

$$\Rightarrow (u^3)' \in C^0 \Rightarrow u^3 \in C^1 \stackrel{?}{\Rightarrow} u \in C^1$$

$$u^3 \in C^1 \Rightarrow u \in C^0 \Rightarrow u \in C^1$$

$\Downarrow$

$$u^3 \in \text{Lip} \Rightarrow u \in \frac{1}{3}\text{-Hölder continuous}$$

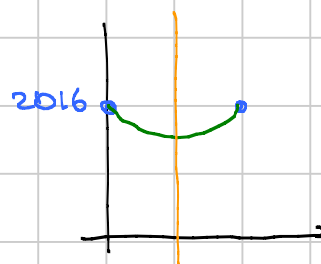
General fact Assume that $f: [a,b] \rightarrow \mathbb{R}$ is continuous and f^α is Lipschitz for some exponent $\alpha > 1$. Then f is $\frac{1}{\alpha}$ -Hölder continuous

In conclusion, in the example $u \in C^{\uparrow 1, 1/3}([0,1])$
 $\uparrow$ $u \in C^1$ $\uparrow$ $u \in \frac{1}{3}\text{-Hölder cont.}$

Remark $u \in C^\infty(A)$ where $A = \{x \in (0,1) : \dot{u}(x) \neq 0\}$

$$\ddot{u} = \frac{2u^7}{3\dot{u}^3} \text{ in } A \text{ no bootstrap argument.}$$

In the example we are sure that $\dot{u}(x) = 0$ somewhere in $(0,1)$ because of DBC.



Exercises (?)

- Prove that $u(x) = u(1-x)$
- Prove that $u \notin C^{1,\alpha}([0,1])$ if $\alpha > \frac{1}{3}$.

Uniqueness is trivial by convexity

Example

$$\min \left\{ \int_0^2 (1+u^2) \dot{u}^6 : u(0)=0, u(2)=2016 \right\}$$

$$L(x, s, p) = (1+s^2) p^6$$

$L(x, s, p) \geq p^6$ for every admissible $(x, s, p) \Rightarrow$ compactness

$$F(u_n) \leq M \Rightarrow \|u_n\|_{L^6} \leq M_1 \Rightarrow \begin{aligned} & \text{(up to subsequences)} \\ & u_n \rightarrow u_\infty \text{ in } L^1 \text{ or } L^6 \\ & u_n \rightarrow u_\infty \text{ unif.} \end{aligned}$$

We need LSC

$$F(u_n) = \int_0^1 \underbrace{g(u_n)}_{(1+u_n^2)} \underbrace{\psi(\dot{u}_n)}_{\dot{u}_n^6} = \int_0^1 g(u_\infty) \psi(\dot{u}_\infty) + \int_0^1 \underbrace{(g(u_n) - g(u_\infty)) \psi(\dot{u}_n)}_{\substack{\text{small in } L^\infty \\ \text{norm when } u \\ \text{is large}}} \psi(\dot{u}_n)$$

As for the first term:

$$\int_0^1 g(u_\infty) \psi(\dot{u}_n) \geq \int_0^1 g(u_\infty) \psi(\dot{u}_\infty) + \int_0^1 g(u_\infty) \psi'(\dot{u}_\infty) (\dot{u}_n - \dot{u}_\infty)$$

we used that $\psi(b) \geq \psi(a) + (b-a)\psi'(a)$
and $g(s) \geq 0$

Now we can take the limit on the left and on the RHS.

If we want to be precise, there are two possibilities

→ again approximate ψ with the lemma so that $\psi'(u_\infty) \in L^\infty$

→ $\psi(u_\infty) = 6u_\infty^5 \in L^{\frac{6}{5}}((0,2))$ and $p=6$ so $q=\frac{6}{5}$

$$\text{So } \int_a^b \underbrace{g(u_\infty) \psi'(u_\infty)}_{\substack{\text{test function} \\ \text{in } L^{6/5}}} \underbrace{(u_n - u_\infty)}_{\substack{\rightarrow 0 \\ \text{weakly in } L^6}} \rightarrow 0$$

⇒ once again conclusion by direct method.

Example $\int_0^l u^4 - |u|^\alpha$ with $u(0) = u(1) = 0$

In this case

→ existence by direct method if $\alpha < 4$

→ non-existence (and actually $\inf = -\infty$) if $\alpha > 4$

→ If $\alpha = 4$ we have existence or non-existence depending on l (existence for small l).

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