

Motivational example

$$\min \left\{ \int_0^1 (u^4 + u^8) dx : u(0) = u(1) = 2016 \right\}$$

Main issues : $\rightarrow$ non quadratic growth wrt p
 $\rightarrow L_{pp} \geq 0$ but not strictly positive (affects regularity)

Direct method : let u_n be a min. sequence. We know that $\|u_n\|_{L^4(0,1)}$ is bounded. We would like to deduce that

$$\begin{aligned} u_n &\rightarrow u_\infty \\ u_n &\rightarrow u_\infty \end{aligned}$$

in some sense (up to subsequences)

Rank $\|u_n\|_{L^4} \leq M \Rightarrow \|u_n\|_{L^2} \leq M_1$

Proof. $x^2 \leq x^4 + 1 \quad \forall x \in \mathbb{R}$ and therefore

$$\int_0^1 u_n^2 \leq \int_0^1 u_n^4 + \int_0^1 1$$

Therefore $u_n \rightarrow v_\infty$ in $L^2(0,1)$ (up to subsequences)

Combining with DBC, we obtain that $u_n \rightarrow u_\infty$ unif.
and $v_\infty = u_\infty$.

Questions : ① $v_\infty \in L^4(0,1)$?

② $\liminf_{n \rightarrow \infty} \int_0^1 u_n^4 \geq \int_0^1 v_\infty^4$?

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Def. (weak convergence in L^p). Let $\{u_n\} \subseteq L^p((a,b))$ and let $u_\infty \in L^p((a,b))$, with $p \in [1, +\infty)$.

We say that

$$u_n \rightharpoonup u_\infty \quad \text{in } L^p((a,b))$$

if

$$\int_a^b u_n(x) \varphi(x) dx \rightarrow \int_a^b u_\infty(x) \varphi(x) dx \quad \forall \varphi \in L^q((a,b))$$

where $\frac{1}{p} + \frac{1}{q} = 1$ ($q = +\infty$ if $p = 1$).

Remark Since $L^\infty((a,b)) \subseteq L^q((a,b))$ for every admissible q , then weak convergence in $L^1((a,b))$ is the weakest.

Weak convergence is easier when p is smaller.
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Theorem (LSC) Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Let $v_n \rightharpoonup v_\infty$ in $L^1((a,b))$ (weakest possible sense).

Then

$$\liminf_{n \rightarrow +\infty} \int_a^b \psi(v_n) dx \geq \int_a^b \psi(v_\infty) dx$$

Proof in the easy case $\psi \in C^1(\mathbb{R})$ and ψ' bounded.

Use the inequality

$$\psi(b) \geq \psi(a) + (b-a) \psi'(a) \quad \forall (a,b) \in \mathbb{R}^2$$

and obtain that

$$\int_a^b \psi(v_n) \geq \int_a^b \psi(v_\infty) + \underbrace{\int_a^b (v_n - v_\infty) \psi'(v_\infty)}_{\substack{\downarrow 0 \text{ as } n \rightarrow +\infty \text{ because} \\ v_n \rightharpoonup v_\infty \text{ in } L^1}} \quad \psi'(v_\infty) \in L^\infty((a,b))$$

Take liminf on both sides.

Lemma (Approx. of convex functions from below)

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Then there exists a family of functions $\psi_\varepsilon: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

(i) ψ_ε is convex for every $\varepsilon > 0$

(ii) $\psi_\varepsilon \in C^1(\mathbb{R})$ " " " and ψ'_ε is bounded

(iii) $\psi_\varepsilon(x) \leq \psi(x)$ " " " and every $x \in \mathbb{R}$

(iv)

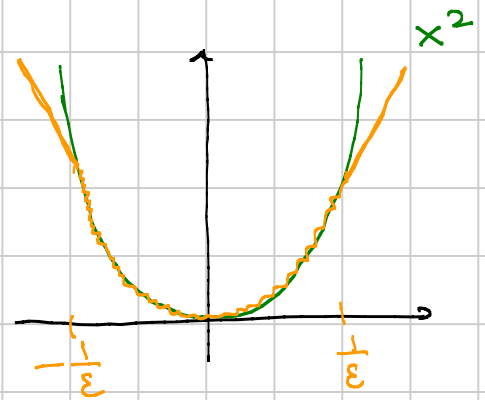
$$\lim_{\varepsilon \rightarrow 0^+} \psi_\varepsilon(x) = \psi(x) \quad \forall x \in \mathbb{R}$$

(v) the family is increasing, namely

$$\psi_{\varepsilon_1}(x) \leq \psi_{\varepsilon_2}(x) \quad \forall x \in \mathbb{R} \quad \forall 0 < \varepsilon_2 \leq \varepsilon_1$$

Proof. Very easy if $\psi \in C^1$, in which case it is enough to truncate ψ' outside $[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]$

If $\psi \notin C^1$, we need to regularize ψ in a first step.



Proof of LSC given the Lemma Fix $\varepsilon > 0$. For every $n \in \mathbb{N}$:

$$\int_a^b \psi(v_n) dx \geq \int_a^b \psi_\varepsilon(v_n) dx$$

$\uparrow$
(ccc)

From the easy part we deduce that

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(v_n) dx \geq \int_a^b \psi_\varepsilon(v_\infty) dx$$

Since ψ_ε is an increasing family, from Beppo Levi theorem we deduce that

$$\int_a^b \psi(v_\infty) dx = \sup_{\varepsilon > 0} \int_a^b \psi_\varepsilon(v_\infty) dx.$$

Back to motivating example $i_n \rightarrow i_\infty$ in L^2 weak
 $u_n \rightarrow u_\infty$ unif.

Therefore $i_n \rightarrow i_\infty$ in L^2 weak

LSC Theorem $\Rightarrow$

$$\liminf_{n \rightarrow \infty} \int_0^1 i_n^4 + u_n^8 \geq \int_0^1 i_\infty^4 + u_\infty^8$$

Direct.

Direct method $\Rightarrow$ existence of minimizers.

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Growth conditions Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$.

We say that ψ has SUPERLINEAR GROWTH if

$$\lim_{|p| \rightarrow \infty} \frac{\psi(p)}{|p|} = +\infty$$

We say that ψ has (at least) quadratic growth if

$$\liminf_{|p| \rightarrow \infty} \frac{\psi(p)}{p^2} > 0$$

or equivalently if there exist $A > 0$ and $B \in \mathbb{R}$ s.t.

$$\psi(p) \geq A|p|^2 - B \quad \forall p \in \mathbb{R}.$$

Theorem (Compactness) Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a function with superlinear growth. Let $v_n: [a,b] \rightarrow \mathbb{R}$ be a sequence s.t.

$$\int_a^b \psi(v_n) dx \leq M \leftarrow \text{indep. of } n$$

Then there exist $v_\infty \in L^2((a,b))$ and $n_k \nearrow \infty$ s.t.

$$v_{n_k} \rightharpoonup v_\infty \text{ weakly in } L^2((a,b))$$

Proof in the special case of quadratic growth

We know that

$$\psi(p) \geq Ap^2 - B \quad \forall p \in \mathbb{R}$$

and therefore

$$p^2 \leq \frac{1}{A} (\psi(p) + B)$$

so that

$$\begin{aligned} \int_a^b v_n^2 &\leq \frac{1}{A} \int_a^b \psi(v_n) + \frac{B}{A} (b-a) \\ &\leq M_1 \end{aligned}$$

Up to subsequences $v_n \rightharpoonup v_\infty$ in L^2 weak, and therefore also in L^1 weak.

Remark $\psi(p) = |p| \log(1+|p|)$ has superlinear growth.

Prop. Assume that $\|v_n\|_{L^p} \leq M$ for some $p > 1$.
Assume that

$$v_n \rightharpoonup v_\infty \text{ in } L^1((a,b))$$

Then $v_\infty \in L^p$ and

$$v_n \rightharpoonup v_\infty \text{ in } L^p((a,b))$$

Proof. $v_\infty \in L^p$ follows from the LSC result.

Let $\varphi \in L^q((a,b))$, and let $\varepsilon > 0$. Take any function $\varphi_\varepsilon \in L^\infty$ s.t.

$$\|\varphi - \varphi_\varepsilon\|_{L^q} \leq \varepsilon$$

Then

$$\int_a^b (v_n - v_\infty) \varphi = \int_a^b (v_n - v_\infty) \varphi_\varepsilon + \int_a^b (v_n - v_\infty) (\varphi - \varphi_\varepsilon)$$

and therefore

$$\begin{aligned}
 \left| \int_a^b (v_n - v_\infty) \varphi \right| &\leq \underbrace{\left| \int_a^b (v_n - v_\infty) \varphi_\varepsilon \right|}_{\substack{\downarrow \\ 0}} + \underbrace{\int_a^b |v_n - v_\infty| \cdot |\varphi - \varphi_\varepsilon|}_{\text{Hölder's inequality}} \\
 &\leq \|v_n - v_\infty\|_{L^p} \|\varphi - \varphi_\varepsilon\|_{L^q} \\
 &\leq \underbrace{\|v_\infty\| + \|v_n\|}_{\leq 2M} \underbrace{\|\varphi - \varphi_\varepsilon\|_{L^q}}_{\leq \varepsilon} \\
 &\leq 2M\varepsilon
 \end{aligned}$$

Therefore

$$\limsup_{n \rightarrow \infty} \left| \int_a^b (v_n - v_\infty) \varphi \right| \leq 2M\varepsilon$$

Since ε is arbitrary the limit of the LHS is 0.

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Remark when $\{v_n\}$ is bounded in L^p , it is enough to test weak convergence against a dense subset of test functions of φ .

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