

Integral functionals with multiple integrals

Domain $\Omega \subseteq \mathbb{R}^m$ (ball or square in $\mathbb{R}^2$). The unknown

$$u: \Omega \rightarrow \mathbb{R} \quad (m \geq 1)$$

$$F(u) := \int_{\Omega} L(x, u(x), \nabla u(x)) dx$$

↑ multiple integral

$$L: \underbrace{\Omega}_x \times \underbrace{\mathbb{R}}_s \times \underbrace{\mathbb{R}^m}_p$$

Example (Dirichlet integral) $F(u) = \int_{\Omega} |\nabla u(x)|^2 dx$

Boundary conditions: prescribing the values of $u(x)$ for $x \in \partial\Omega$

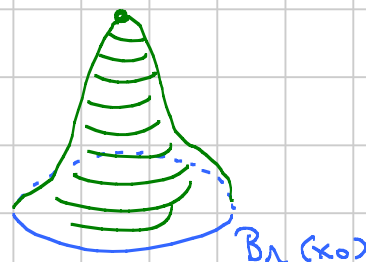
Remark Pointwise conditions such as $u(a) = A$ with $a \in \bar{\Omega}$ make no sense for $F(u)$.

Roughly speaking, this is because $\forall B_r(x_0)$

$$\inf \{ F(u) : u(x_0) = 1, u(x) \equiv 0 \text{ at } \partial B_r(x_0) \} = 0$$

$$u \in C^1(B_r(x_0))$$

(Exercise: find "optimal profiles")



If $\Omega \subseteq \mathbb{R}^m$, the only conditions which make sense are on sets of dim $m-1$, for example $\partial\Omega$.

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$$\min \left\{ \int_{\Omega} |\nabla u|^2 dx : u(x) = \underset{\substack{\uparrow \\ \text{given}}}{\varphi}(x) \quad \forall x \in \partial\Omega \right\}$$

Indirect method Assume $u_0 \in \arg\min$

$$F(u_0 + tv)$$

$\uparrow v \in C^1(\Omega)$ and $v \equiv 0$ at $\partial\Omega$



$$\begin{aligned} &= \int_{\Omega} |\nabla(u_0 + tv)|^2 = \int_{\Omega} |\nabla u_0 + t \nabla v|^2 \\ &= \int_{\Omega} |\nabla u_0|^2 + t^2 \int_{\Omega} |\nabla v|^2 + 2t \int_{\Omega} \langle \nabla u_0, \nabla v \rangle \end{aligned}$$

$$\delta F(u_0, v) = 2 \int_{\Omega} \langle \nabla u_0, \nabla v \rangle dx$$

First form of Euler Equation

In order to integrate by parts, we need Gauss - Green formula

$$\int_{\Omega} f \operatorname{div} \vec{E} = \underbrace{\int_{\partial\Omega} \langle \vec{E}, \vec{n} \rangle f d\sigma}_{(n-1)\text{-dim. integral}} - \int_{\Omega} \langle \nabla f, \vec{E} \rangle dx$$

Let us apply GG with $f := v$ and $\vec{E} = \nabla u_0$. Obtain that

$$\begin{aligned} \int_{\Omega} \underset{\substack{\downarrow \nabla f}}{\langle \nabla v, \nabla u_0 \rangle} dx &= - \int_{\Omega} \underset{\substack{\downarrow E}}{v \operatorname{div}(\nabla u_0)} + \underbrace{\int_{\partial\Omega} \underset{\substack{\downarrow f}}{v} \langle \nabla u_0, \vec{n} \rangle d\sigma}_{=0 \text{ because } v \equiv 0 \text{ at } \partial\Omega} \\ &= - \int_{\Omega} \Delta u_0 \cdot v \end{aligned}$$

where $\Delta u = u_{xx} + u_{yy}$

Therefore, the second form of Euler equ is

$$\boxed{\int_{\Omega} \Delta u_0 \cdot v \, dx = 0} \quad \forall v \text{ s.t. } v \equiv 0 \text{ at } \partial\Omega$$

$$\text{FLCV} \Rightarrow \begin{cases} \Delta u_0 = 0 & \text{(partial differential equ)} \\ u_0 = \varphi & \text{at } \partial\Omega \quad \text{(DBC)} \end{cases} \quad \left. \vphantom{\begin{cases} \Delta u_0 = 0 \\ u_0 = \varphi \end{cases}} \right] \begin{array}{l} \text{Euler equ.} \\ \text{in diff.} \\ \text{form} \end{array}$$

Assume we are able to solve. Then u_0 is the unique minimum point (by convexity)

Any other competitor w can be written as $u_0 + v$ with $v \in V$ and

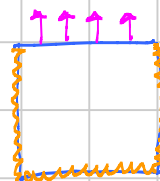
$$F(w) = F(u_0 + v) = \underbrace{\int_{\Omega} |\nabla u_0|^2}_{F'(u_0)} + \underbrace{\int_{\Omega} |\nabla v|^2}_{\geq 0} + 2 \underbrace{\int_{\Omega} \langle \nabla u_0, \nabla v \rangle}_{=0 \text{ because of Euler equ.}}$$

$\underbrace{\quad}_{=0} \quad \underbrace{\quad}_{=0} \quad \underbrace{\quad}_{\Rightarrow v \equiv 0}$

Example 2 $F(u) = \text{Dirichlet integral}$ $\Omega = \text{square}$

$$\min \{ F(u) : u = \varphi \text{ on three sides} \}$$

$\uparrow$
given



As before $\delta F(u_0, v) = \int_{\Omega} \langle \nabla u_0, \nabla v \rangle = 0 \quad \forall v \equiv 0 \text{ on 3 sides}$

Integrate by parts (G.G.):

$$\int_{\Omega} \langle \nabla u_0, \nabla v \rangle \, dx = - \int_{\Omega} \Delta u_0 \cdot v \, dx + \int_{\partial\Omega} v \langle \nabla u_0, \vec{n} \rangle \, d\sigma$$

$\uparrow$
outer normal

1st step We limit to $v \equiv 0$ on the whole $\partial\Omega$
 $\leadsto \Delta u_0 = 0$ in Ω

2nd step Now we know that

$$\int_{\partial\Omega} v \langle \nabla u_0, \vec{n} \rangle d\sigma = 0 \quad \forall v \text{ admissible}$$

$\uparrow$ actually on 1 side only

FLCV on the side and deduce that $\langle \nabla u_0, \vec{n} \rangle = 0$ on the side # 4

$$\langle \nabla u, \vec{n} \rangle = \frac{\partial u}{\partial n}$$

$\uparrow$ directional derivative of u in the direction $\vec{n}$

At the end u_0 has to solve

$$\begin{cases} \Delta u = 0 & \text{PDE} \\ u \equiv \varphi & \text{on 3 sides} & \text{DBC} \\ \frac{\partial u}{\partial n} = 0 & \text{at 4th side} & \text{NBC} \end{cases}$$

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What about the general case

$$F(u) = \int_{\Omega} L(x, u, \nabla u) dx$$

$$F(u+tv) =: \varphi(t)$$

$$\varphi'(0) = \frac{d}{dt} \int_{\Omega} L(x, u+tv, \nabla u + t\nabla v) dx \Big|_{t=0}$$

$$= \int_{\Omega} L_s(x, u, \nabla u) v + \langle \nabla_p L(x, u, \nabla u), \nabla v \rangle = \delta F(u, v)$$

$\uparrow$
gradient of L wrt p variable

The second term can be integrated by parts ...

$$\int_{\Omega} L_s(x, u, \nabla u) v - \operatorname{div} \nabla_p L(x, u, \nabla u) \cdot v + \int_{\partial\Omega} v \langle \nabla_p L(x, u, \nabla u), \vec{n} \rangle d\sigma$$

As before, we restrict to $v \equiv 0$ at $\partial\Omega$

$$\operatorname{div} (\nabla_p L(x, u, \nabla u)) = L_s(x, u, \nabla u)$$

Euler equ. in diff. form

$$\langle \nabla_p L(x, u, \nabla u), \vec{n} \rangle = 0$$

in the points of $\partial\Omega$ where DBC where not prescribed.

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What about the direct method?

Sobolev spaces can be defined exactly as in dim 1 (W-H)

In dim. $n \geq 2$ Sobolev functions are not nec. continuous, and in particular H^1 functions are not continuous.

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