

Lagrange Multipliers

Back to calculus : $x_0 \in \mathbb{R}^n$, $r > 0$, $F: B_r(x_0) \rightarrow \mathbb{R}$

$$G: B_r(x_0) \rightarrow \mathbb{R}$$

$$V := \{x \in B_r(x_0) : G(x) = 0\}$$

$$\min \{F(x) : x \in V\}$$

Lagrange multipliers Assume F and G of class C^1 in $B_r(x_0)$

Assume that x_0 is a solution to the min. prob.

Then at least one of the following options is true

(i) $\nabla G(x_0) = 0$ (x_0 is a stationary point for $G(x)$)

(ii) $\exists \lambda \in \mathbb{R}$ s.t. x_0 is a stationary point of $\Phi(x) = F(x) - \lambda G(x)$
 $(\nabla F(x_0) = \lambda \nabla G(x_0))$

Remark If $V = \{x \in B_r(x_0) : G_1(x) = \dots = G_k(x) = 0\}$

then the options are

(i) $\exists (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \setminus \{0\}$ s.t. x_0 is a stationary point of
 $\lambda_1 G_1(x) + \dots + \lambda_k G_k(x)$

(ii) $\exists (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ s.t. x_0 is a stationary point of

$$\Phi(x) := F(x) - \lambda_1 G_1(x) - \dots - \lambda_k G_k(x)$$

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Back to calculus of variations

Assume that $F(u)$ and $G(u)$ are two functionals defined in some affine space X with reference vector space V .
 let

$$W = \{u \in X : G(u) = 0\}$$

Theorem Assume

$$u_0 \in \operatorname{argmin} \{ F(u) : u \in W \}$$

Then at least one of the following options is true:

(i) $\delta G(u_0, v) = 0$ for every $v \in V$ (provided that δG exists)

(ii) $\exists \lambda \in \mathbb{R}$ s.t. $\delta F(u_0, v) = \lambda \delta G(u_0, v)$ for every $v \in V$,
namely

$$\delta \Phi(u_0, v) = 0 \quad \forall v \in V$$

where

$$\Phi(u) := F(u) - \lambda G(u).$$

Two strategies : $\rightarrow$ via implicit function thm.

$\rightarrow$ via inverse function thm

Proof 1 If (i) holds true, there is nothing to prove.

So we can assume that $\delta G(u_0, w) \neq 0$ for some $w \in V$.

Define

$$f(t, s) := F(u_0 + tv + sw) \quad v \in V$$

$$g(t, s) := G(u_0 + tv + sw)$$

We have that

$$g_t(0, 0) = \delta G(u_0, v)$$

$$g_s(0, 0) = \delta G(u_0, w) \neq 0$$

$$g(0, 0) = 0$$

Implicit function theorem $\Rightarrow \exists \delta > 0$ and $s : (-\delta, \delta) \rightarrow \mathbb{R}$

such that $s(0) = 0$ and

$$g(t, s(t)) = 0 \quad \forall t \in (-\delta, \delta)$$

and

$$s'(t) = - \frac{g_t(t, s(t))}{g_s(t, s(t))} \rightsquigarrow s'(0) = - \frac{\delta G(u_0, v)}{\delta G(u_0, w)}$$

Now consider $\varphi(t) := f(t, s(t)) = F(\underbrace{u_0 + tv + s(t)w}_{\in W})$
defined for $t \in (-\delta, \delta)$.

We know that $t=0$ is a min point for φ and therefore $\varphi'(0) = 0$

$$\varphi'(t) = f_t(t, s(t)) + f_s(t, s(t)) s'(t)$$

$$\varphi'(0) = f_t(0, 0) + f_s(0, 0) s'(0)$$

$$= \delta F(u_0, v) + \underbrace{\delta F(u_0, w)}_{\lambda} \left(- \frac{\delta G(u_0, v)}{\delta G(u_0, w)} \right)$$

$$\Rightarrow \underbrace{\delta F(u_0, v)}_0 = \underbrace{\lambda}_{\neq 0} \underbrace{\delta G(u_0, v)}_0 \quad \forall v \in V$$

Proof 2 Let $f(t, s)$ and $g(t, s)$ be as before. Let $w \in V$ be as before, namely $g_s(0, 0) = \delta G(u_0, w) \neq 0$. Let us consider

$$\varphi(t, s) := (f(t, s) - \min \{F(u) : u \in W\}, g(t, s))$$

$$\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

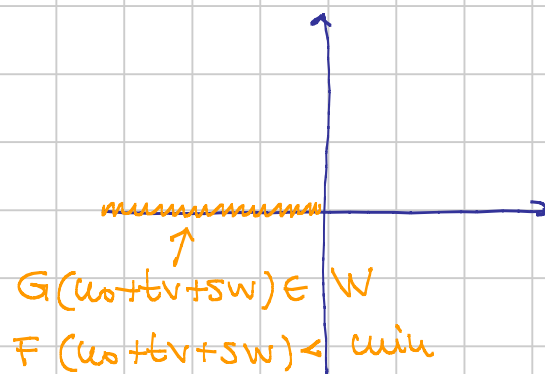
Main point: the image of φ does not contain the negative x axis.

Let us consider the matrix

$$J\varphi(0, 0) = \begin{pmatrix} f_t(0, 0) & f_s(0, 0) \\ g_t(0, 0) & g_s(0, 0) \end{pmatrix}$$

$$= \begin{pmatrix} \delta F(u_0, v) & \delta F(u_0, w) \\ \delta G(u_0, v) & \delta G(u_0, w) \end{pmatrix}$$

$$\varphi(0, 0) = 0$$



Assume that $\text{rank } J\varphi(0, 0) = 2$. Then by the inverse function

then the image of φ contains a neighborhood of the origin, which is absurd.

Therefore $\text{rank} = 1$, and hence $\text{Det} = 0$

$$\delta F(u_0, v) \underbrace{\delta G(u_0, w)}_{\text{--- } 0 \text{ --- } 0 \text{ ---}} = \delta F(u_0, w) \delta G(u_0, v)$$

Rank (exercise) Generalize the 2nd proof to the case of k constraints.

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Example 1

$$F(u) = \int_0^l \dot{u}^2 dx$$

$$G(u) = \int_0^l u^2 dx - 1$$

$$X = \{u \in C^1([0, l]) : u(0) = u(l) = 0\}$$

Direct method $\Rightarrow$ existence of a min point in $H^1((0, l))$
(only need to check that the constraint is stable wrt the notion of convergence)

By Lagrange method there are 2 possibilities

$$(i) \delta G(u_0, v) = 0 \leadsto \text{Euler eqn of } G \leadsto 0 = 2u$$

$$(L_F)' = L_S$$

$$\Rightarrow u_0 \equiv 0 \Rightarrow \text{does not satisfy } G(u) = 0$$

(ii) u_0 satisfies the Euler equation of the functional with

$$L(x, s, p) = p^2 - \lambda s^2$$

$$(F - \lambda G)$$

Lagrange multiplier

$$(2\dot{u})' = -2\lambda u \leadsto \begin{cases} \ddot{u} = -\lambda u \\ u(0) = 0 \\ u(l) = 0 \end{cases}$$

What about sign of λ ? Multiply by u :

$$\ddot{u}u = -\lambda u^2 \quad \leadsto \text{integrate}$$

$$\int \ddot{u}u = -\lambda \int u^2 = -\lambda$$

$$= -\int \dot{u}^2 + \underbrace{[\dot{u}u]_0^l}_0 \quad \Rightarrow \lambda > 0$$

We can solve the equation and obtain

$$u(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$\overset{0}{\text{because}}$

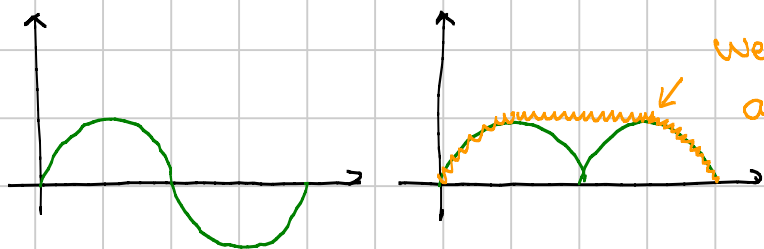
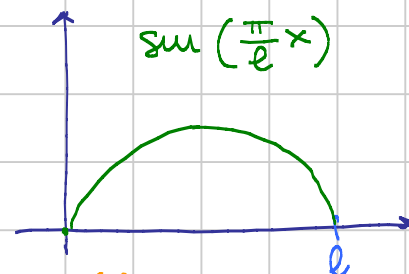
$$u(0) = 0$$

$$\sin(\sqrt{\lambda}l) = 0 \leadsto \sqrt{\lambda} = \frac{k\pi}{l}$$

$$u(x) = c \sin\left(\frac{k\pi x}{l}\right) \leadsto \text{use } \int u^2 = 1 \text{ and find } c$$

$\leadsto$ compute $\int \dot{u}^2$ and find that the minimum is for $k=1$
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Remark Is it possible to see that $k=1$ is better than $k > 1$?



we save on $\int \dot{u}^2$
and $\int u^2$ is larger, so that
we can multiply by
a constant less than 1