

WEIERSTRASS NECESSARY CONDITION

$$\boxed{\text{No SLM}} \Rightarrow \boxed{(E) + (L) + [(L^+) \Rightarrow (J)] + (W)}$$

$\Downarrow$
 no WLM

Remains to prove $\text{no SLM} \Rightarrow (W)$

$$(W) \quad E(x, u_0(x), \dot{u}_0(x), q) \geq 0 \quad \forall x \in [a, b], \quad \forall q \in \mathbb{R}$$

Proof. We prove actually that

$$E(x, u_0(x), \dot{u}_0(x), \dot{u}_0(x) + q) \geq 0 \quad \forall x \in (a, b), \quad \forall q \in \mathbb{R}$$

Let $x_0 \in (a, b)$ be any point. Let $q \in \mathbb{R}$. Consider $v_\varepsilon(x)$ as follows:
 let $\eta > 0$ be any pos. real number

If $\varepsilon > 0$ is small enough, then $v_\varepsilon(x)$ is an admissible variation (0 at the boundary) and therefore



$$F(u_0 + v_\varepsilon) \geq F(u_0) \quad \text{by defn. of SLM}$$

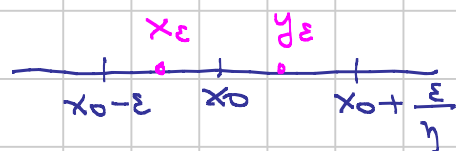
$$F(u_0 + v_\varepsilon) - F(u_0) \geq 0$$

$$\begin{aligned}
 \text{LHS} &= \int_{x_0 - \varepsilon}^{x_0} L(x, u_0(x) + v_\varepsilon(x), \dot{u}_0(x) + q) - L(x, u_0(x), \dot{u}_0(x)) \, dx \\
 &\quad + \int_{x_0}^{x_0 + \frac{\varepsilon}{\eta}} L(x, u_0(x) + v_\varepsilon(x), \dot{u}_0(x) - \eta q) - L(x, u_0(x), \dot{u}_0(x)) \, dx \geq 0
 \end{aligned}$$

Let us divide by $\varepsilon > 0$ and let us apply mean value theorem for integrals

$$L(x_\varepsilon, u_0(x_\varepsilon) + v_\varepsilon(x_\varepsilon), \dot{u}_0(x_\varepsilon) + q) - L(x_\varepsilon, u_0(x_\varepsilon), \dot{u}_0(x_\varepsilon)) \\ + \frac{1}{\eta} L(y_\varepsilon, u_0(y_\varepsilon) + v_\varepsilon(y_\varepsilon), \dot{u}_0(y_\varepsilon) - \eta q) - \frac{1}{\eta} L(y_\varepsilon, u_0(y_\varepsilon), \dot{u}_0(y_\varepsilon)) \geq 0$$

Let us send $\varepsilon \rightarrow 0^+$



$$L(x_0, u_0(x_0), \dot{u}_0(x_0) + q) - L(x_0, u_0(x_0), \dot{u}_0(x_0)) \\ + \boxed{\frac{L(x_0, u_0(x_0), \dot{u}_0(x_0) - \eta q) - L(x_0, u_0(x_0), \dot{u}_0(x_0))}{\eta}} \geq 0 \\ \downarrow \\ -q L_p(x_0, u_0(x_0), \dot{u}_0(x_0))$$

Pass to the limit as $\eta \rightarrow 0^+$ and obtain

$$E(x_0, u_0(x_0), \dot{u}_0(x_0), \dot{u}_0(x_0) + q) \geq 0$$

— 0 — 0 —

Sufficient condition

$$(E) + (L^+) + (J^+) + (W^+) \Rightarrow u_0 \text{ is SLM.}$$

Proved if we show that

$$(E) + (L^+) + (J^+) \Rightarrow (F)$$

IMBEDDING THEOREM

Main idea $u_0(x)$ is the solution to Euler equ

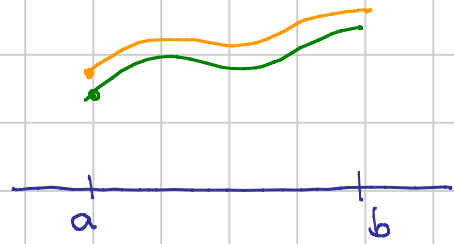
$$\frac{d}{dx} L_p(x, u, \dot{u}) = L_s(x, u, \dot{u})$$

with initial conditions $u_0(a)$ and $\dot{u}_0(a)$ given

Let us consider the same equation with initial conditions

$$u(a) = u_0(a) + \varepsilon \alpha$$

$$\dot{u}(a) = \dot{u}_0(a) + \varepsilon \beta$$



Hope: finding α and β in such a way that solutions to the approx. prob. form the required stratification.

Let $u(\varepsilon, x)$ denote the solution. Hope is ok if we show that

$$u_\varepsilon(0, x) \geq m > 0 \quad \text{for every } x \in [a, b]$$

This guarantees that $\exists \varepsilon_0 > 0$ s.t.

$$u_\varepsilon(\varepsilon, x) > 0 \quad \text{for every } (\varepsilon, x) \in (-\varepsilon_0, \varepsilon_0) \times [a, b]$$

We have reduced the problem to computing the derivative of the solution w.r.t initial conditions.

Surprise: $u_\varepsilon(0, x) =: w(x)$ is the solution to (JDE) with initial data $w(a) = \alpha$, $\dot{w}(a) = \beta$ and we know that

$(L^+) + (J^+) \Rightarrow (J^{++})$, namely the existence of a solution to (JDE) which is strictly positive in $[a, b]$.

Remains to prove that $w(x)$ solves (JDE)

Sketch of the proof

$$w_\varepsilon(x) := \frac{u(\varepsilon, x) - u(0, x)}{\varepsilon} = \frac{u(\varepsilon, x) - u_0(x)}{\varepsilon}$$

For every $\varepsilon > 0$:

$$[L_p(x, u(\varepsilon, x), \dot{u}(\varepsilon, x))]^\dagger = L_s(x, u(\varepsilon, x), \dot{u}(\varepsilon, x))$$

$$[L_p(x, u_0(x), \dot{u}_0(x))]^\dagger = L_s(x, u_0(x), \dot{u}_0(x)) \quad \leftarrow \varepsilon = 0$$

Subtract

$$[L_p(x, u(\varepsilon, x), \dot{u}(\varepsilon, x)) - L_p(x, u_0(x), \dot{u}_0(x))]^\dagger = L_s(\dots) - L_s(\dots)$$

$$[L_{ps}(\dots)(u(\varepsilon, x) - u_0(x)) + L_{pp}(\dots)(\dot{u}(\varepsilon, x) - \dot{u}_0(x))]^\dagger =$$

$$L_{ss}(\dots)(u(\varepsilon, x) - u_0(x)) + L_{sp}(\dots)(\dot{u}(\varepsilon, x) - \dot{u}_0(x))$$

Divide by ε :

$$[L_{ps}(\dots) w_\varepsilon + L_{pp}(\dots) \dot{w}_\varepsilon]^\dagger = L_{ss}(\dots) w_\varepsilon + L_{sp}(\dots) \dot{w}_\varepsilon$$

Take the limit as $\varepsilon \rightarrow 0^+$

$$[L_{ps}(\dots) w + L_{pp}(\dots) \dot{w}]^\dagger = L_{ss}(\dots) w + L_{sp}(\dots) \dot{w}$$

$\uparrow \quad \quad \quad \nearrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow$
 $(x, u_0(x), \dot{u}_0(x))$

$$[B(x) w + C(x) \dot{w}]^\dagger = A(x) w + B(x) \dot{w} \quad \leftarrow \text{(JDE)}$$

— 0 — 0 —

Simpler example

$$\begin{cases} \dot{u} = f(t, u) \\ u(0) = \varepsilon \end{cases}$$

Let $u(\varepsilon, x)$ denote the solution. Let $v(x) := u_\varepsilon(0, x)$.

What is the equation solved by v ?

$$\dot{u}(\varepsilon, t) = f(t, u(\varepsilon, t))$$

$$\dot{u}(0, t) = f(t, u(0, t))$$

$$\frac{\dot{u}(\varepsilon, t) - \dot{u}(0, t)}{\varepsilon} = \frac{1}{\varepsilon} [f(t, u(\varepsilon, t)) - f(t, u(0, t))]$$

$$\left[\frac{u(\varepsilon, t) - u(0, t)}{\varepsilon} \right]' = \frac{1}{\varepsilon} [f_u(t, c(\varepsilon, t)) (u(\varepsilon, t) - u(0, t))]$$

$$\boxed{\dot{v} = f_u(t, u_0) v}$$

$$v(0) = 1$$

— 0 — 0 —