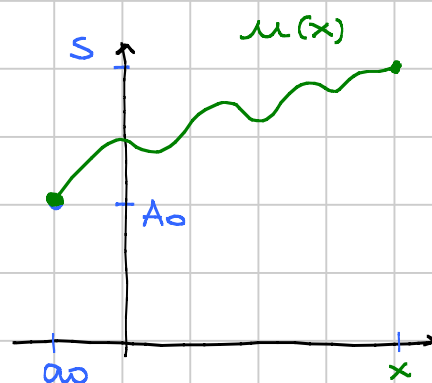


VALUE FUNCTION Assume $L(x, s, p)$ defined in $\mathbb{R}^3$.

Let us fix a "base point" (a_0, A_0)

For every $(x, s) \in \mathbb{R}^2$ let us solve

$$\min \left\{ \int_{a_0}^x L(t, u(t), \dot{u}(t)) dt : \right. \\ \left. u(a_0) = A_0, u(x) = s \right\}$$



Let $V(x, s)$ denote the minimum value.

↑ value function

Theorem Assume that $V(x, s)$ is of class C^1 .

Then it is a verification function, namely the null Lagrangian

$$\hat{L}(x, s, p) := V_s(x, s) p + V_x(x, s)$$

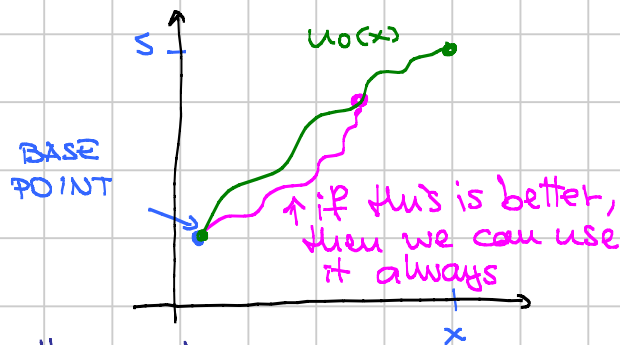
satisfies

$$(ii) \quad L(x, s, p) \geq \hat{L}(x, s, p)$$

$$(iii) \quad L(x, u_0(x), \dot{u}_0(x)) = \hat{L}(x, u_0(x), \dot{u}_0(x)) \quad \text{for every } u_0(x) \\ \text{which realizes the minimum.}$$

Proof. (iii) Assume that

$$\int_{a_0}^x L(t, u_0(t), \dot{u}_0(t)) dt = V(x, s)$$



Then the same relation is true for all points on the graph of u_0 . This proves that

$$\int_{a_0}^x L(t, u_0(t), \dot{u}_0(t)) dt = V(x, u_0(x))$$

If we compute the derivative wrt x we find

$$\begin{aligned} L(x, u_0(x), \dot{u}_0(x)) &= V_x(x, u_0(x)) + V_s(x, u_0(x)) \dot{u}_0(x) \\ &= \hat{L}(x, u_0(x), \dot{u}_0(x)) \end{aligned}$$

which proves (iii).

$$\begin{aligned} V(x+R, s+Rp) &\leq V(x, s) + \\ &\quad + \int_0^R L(x+t, s+tp, p) dt \end{aligned}$$

Dividing by R we obtain

$$\frac{V(x+R, s+Rp) - V(x, s)}{R} \leq \frac{1}{R} \int_0^R L(x+t, s+tp, p) dt$$

when $R \rightarrow 0$ we find that

$$\frac{V_x(x, s) + V_s(x, s)p}{\hat{L}(x, s, p)} \leq L(x, s, p)$$

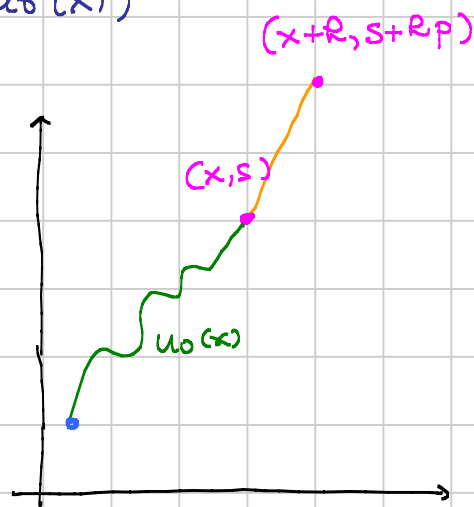
This proves (ii).

— o — o —

In practice: Suppose we need to minimize

$$F(u) = \int_a^b L(x, u, \dot{u}) dx$$

subject to some DBC. Suppose we suspect that some $u_0(x)$ is the minimum. Suppose we think we can solve the problem for any choice of DBC. Then we can compute the value function, and from the value function we can obtain a calibration $\hat{L}(x, s, p)$. If we can check (ii) and (iii), then minimality is proved.



Example with $\{ \int_0^1 \dot{u}^2 : u(0)=0, u(1)=3 \}$

Fix a base point and compute $V(x,s)$ using lines

$$V(x,s) = \left[\frac{(s+3u)}{(x+u)} \right]^2 (x+u)$$

↑
Derivative

$$= \frac{(s+3u)^2}{(x+u)}$$

as in example 2 of previous lecture

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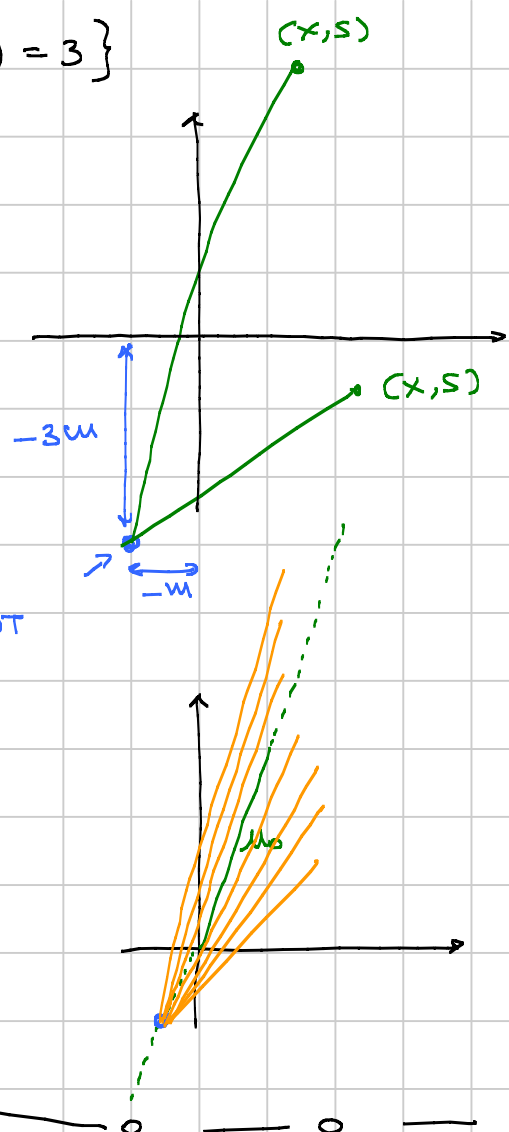
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Develop necessary / sufficient conditions for $u_0(x)$ to be SLM

Def. (WEIERSTRASS EXCESS FUNCTION)

Let $L(x,s,p)$ be the usual Lagrangian

The WEF is defined as

$$E(x,s,p,q) := L(x,s,q) - L(x,s,p) - (q-p)L_p(x,s,p)$$

$$x \in [a,b], (p,q,s) \in \mathbb{R}^3$$

Remark Looks like Taylor expansion of L wrt last variable.

In particular, if $p \rightarrow L(x,s,p)$ is convex for some (x,s) then

$$E(x,s,p,q) \geq 0 \quad \forall (p,q) \in \mathbb{R}^2$$

WEIERSTRASS CONDITIONS

$$(W) \quad E(x, u_0(x), \dot{u}_0(x), q) \geq 0 \quad \forall (x, q) \in [a, b] \times \mathbb{R}$$

$$(W^+) \quad \exists \delta > 0 \text{ s.t. } E(x, s, p, q) \geq 0 \\ \text{for every } (x, s, p, q) \in [a, b] \times \mathbb{R}^3 \text{ s.t. } |s - u_0(x)| \leq \delta \\ |p - \dot{u}_0(x)| \leq \delta$$

Necessary condition

$$u_0 \text{ SLM} \Rightarrow (E) + (L) + [(L^+) \Rightarrow (J)] + (W)$$

Suff. cond.

$$(E) + (L^+) + (J^+) + (W^+) \Rightarrow u_0 \text{ SLM}$$

Road map for suff. cond.

$$\textcircled{1} \quad (E) + (L^+) + (J^+) \Rightarrow (F)$$

(EMBEDDING THEOREM)

$$\textcircled{2} \quad (F) \Rightarrow (WRF)$$

↑
Weierstrass representation formula

$$\textcircled{3} \quad (WRF) + (W^+) \Rightarrow u_0 \text{ is SLM}$$

(WRF) If $u(x)$ and $u_0(x)$ have same DBC, then

$$F(u) - F(u_0) = \int_a^b E(\dots) \\ \uparrow \\ \text{suitable points} \\ \text{---} \circ \text{---} \circ \text{---}$$

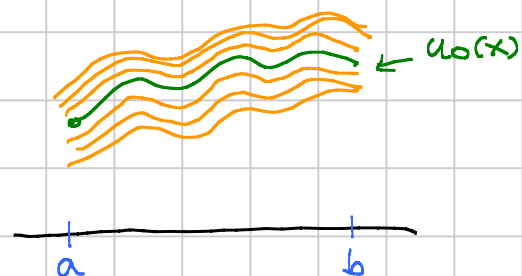
WEIERSTRASS FIELD

Family of solutions to Euler equation which fill an open set around $u_0(x)$.

More precisely, is a function

$$u: (-\varepsilon_0, \varepsilon_0) \times [a, b] \rightarrow \mathbb{R}$$

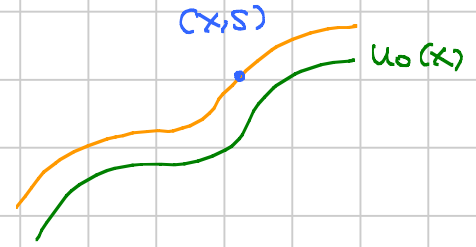
such that $x \rightarrow u(\varepsilon, x)$ is a solution to (E) for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $u_\varepsilon(\varepsilon, x) > 0$ for every (ε, x) admissible



Definition To each WF a SLOPE FUNCTION is associated

Take any (x, s) in the field. There is a unique $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ s.t.

$$u(\varepsilon, x) = s$$



Then the slope function is

$$p(x, s) := \dot{u}(\varepsilon, x) = u_x(\varepsilon, x)$$

The basic relation is

$$u_x(\varepsilon, x) = p(x, u(\varepsilon, x))$$

$$(WRF) \quad F(u) - F(u_0) = \int_a^b E(x, u(x), p(x, u(x)), \dot{u}(x)) dx$$