

"One idiot in a crowd probably remains unnoticed, but a bunch of idiots stands out very well"

[loosely based on De Giorgi]

CALIBRATION

Motivating example 1 $\min \left\{ \int_0^1 \dot{u}^2 dx \mid u(0)=0, u(1)=3 \right\}$

The solution is $u_0(x) = 3x$. New proof: consider

$$p^2 \geq 6p - 9 \quad \forall p \in \mathbb{R}$$

Then

$$F(u) = \int_0^1 \dot{u}^2 dx \geq \int_0^1 6\dot{u} - 9 = 6[u(x)]_0^1 - 9 = 18 - 9 = 9 = F(u_0)$$

Equality holds if and only if there is equality in $\dot{u}^2 \geq 6\dot{u} - 9$ namely iff $\dot{u} \equiv 3$.

Motivating example 2 Same problem, different ineq.

$$p^2 \geq 2 \frac{3m+s}{m+x} p - \left(\frac{3m+s}{m+x} \right)^2 \quad \forall p \in \mathbb{R}, \forall x \geq 0, \forall s \in \mathbb{R}, \forall m > 0$$

As before

$$\begin{aligned} F(u) = \int_0^1 \dot{u}^2 dx &\geq \int_0^1 2 \frac{3m+u(x)}{m+x} \dot{u} - \left(\frac{3m+u(x)}{m+x} \right)^2 \\ &= \left[\frac{(3m+u(x))^2}{m+x} \right]_0^1 = \frac{(3m+3)^2}{m+1} - \frac{9m^2}{m} \\ &= 9(m+1) - 9m = 9 = F(u_0) \end{aligned}$$

Remark Example 1 is the limit of example 2 as $m \rightarrow +\infty$

Motivating example 3 The same as 2, with $m=0$

$$p^2 \geq 2 \frac{s}{x} p - \frac{s^2}{x^2}$$

$$F(u) = \int_0^1 \dot{u}^2 dx \geq \int_0^1 2 \frac{u(x)}{x} \dot{u}(x) - \frac{u(x)^2}{x^2} dx$$
$$= \left[\frac{u(x)^2}{x} \right]_0^1 = 9 = F(u_0)$$

↑
because
 $u(x)^2 = o(x)$ as $x \rightarrow 0$

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NULL LAGRANGIAN Consider a lagrangian of the form

$$\hat{L}(x, s, p) = A(x, s)p + B(x, s)$$

Assume now that $A_x(x, s) = B_s(x, s) \quad \forall (x, s) \in [a, b] \times \mathbb{R}$
Then there exists a function $V(x, s)$ s.t.

$$A(x, s) = V_s(x, s)$$

$$B(x, s) = V_x(x, s)$$

and therefore

$$\int_a^b \hat{L}(x, u(x), \dot{u}(x)) dx = \int_a^b V_s(x, u(x)) \dot{u}(x) + V_x(x, u(x)) dx$$
$$= \left[V(x, u(x)) \right]_a^b$$
$$= V(a, A) - V(b, B)$$

Therefore, the integral functional depends only on DBC of u .

Lemma (Calibration through null Lagrangian)

Let $L(x, s, p)$ be any Lagrangian.

Assume there exist $A(x, s)$ and $B(x, s)$ s.t.

(i) $A_x = B_s$ in $[a, b] \times \mathbb{R}$,

(ii) $L(x, s, p) \geq A(x, s)p + B(x, s)$ in $[a, b] \times \mathbb{R}^2$,

(iii) $L(x, u_0(x), \dot{u}_0(x)) = A(x, u_0(x))\dot{u}_0(x) + B(x, u_0(x))$ in $[a, b]$.

Then $u_0(x)$ is a minimizer of

$$F(u) = \int_a^b L(x, u(x), \dot{u}(x)) dx$$

subject to its DBC.

Proof. As in the examples for every u

$$F(u) = \int_a^b L(x, u(x), \dot{u}(x)) dx \quad (\text{use (ii)})$$

$$\geq \int_a^b [A(x, u(x))\dot{u}(x) + B(x, u(x))] dx \quad (\text{null Lagrangian})$$

$$= [V(x, u(x))]_a^b \quad \text{if } u \text{ and } u_0 \text{ have same DBC}$$

$$= [V(x, u_0(x))]_a^b \quad (\text{null Lagrangian})$$

$$= \int_a^b [A(x, u_0(x))\dot{u}_0(x) + B(x, u_0(x))] dx \quad (\text{use (iii)})$$

$$= \int_a^b L(x, u_0(x), \dot{u}_0(x)) dx$$

$$= F(u_0).$$

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Remark If v_0 is any other minimizers with the same DBC, then v_0 has to satisfy the equality in (iii).

Previous example 1 Quadratic functional

$$L(x, s, p) = A(x)s^2 + 2B(x)sp + C(x)p^2$$

$$V(x, s) = -z(x)s^2 \leadsto \hat{L}(x, s, p) = V_s(x, s)p + V_x(x, s) \\ = \underbrace{-2z(x)sp}_{A(x, s)} - \underbrace{\dot{z}(x)s^2}_{B(x, s)}$$

Condition (i) is trivial

Condition (iii) $L(x, u_0(x), \dot{u}_0(x)) = \hat{L}(x, u_0(x), \dot{u}_0(x))$ is trivial when $u_0(x) \equiv 0$

Condition (ii) $L(x, s, p) \geq \hat{L}(x, s, p)$ is ok if $z(x)$ is the solution to the Riccati eqn in the proof of the suff. condition.

Previous example 2 Assume that $L(x, s, p)$ is convex in (s, p) for every $x \in [a, b]$. Then

$$L(x, s, p) \geq L(x, u_0(x), \dot{u}_0(x)) + (s - u_0(x)) L_s(x, u_0(x), \dot{u}_0(x)) \\ + (p - \dot{u}_0(x)) L_p(\dots) \\ = \hat{L}(x, s, p)$$

Claim: $\hat{L}(x, s, p)$ is a null Lagrangian

$$\hat{L}(x, s, p) = \underbrace{L_p(x, u_0(x), \dot{u}_0(x))}_{A(x, s)} p + \underbrace{L(x, u_0(x), \dot{u}_0(x)) + (s - u_0(x)) L_s(\dots) - \dot{u}_0(x) L_p(\dots)}_{B(x, s)}$$

Let us check (i), (ii), (iii).

- (iii) is trivial
- (ii) is the inequality coming from convexity
- (i) $A_x = B_s \Leftrightarrow \frac{d}{dx} L_p(x, u_0(x), \dot{u}_0(x)) = L_s(\dots) \Leftrightarrow u_0$ satisfies (E)

Back to motivating example 1

$$p^2 \geq 6p - 9$$

$$V(x, s) := 6s - 9x$$

$$\int_a^b \dot{u}(x)^2 dx = V(b, u(b)) - V(a, u(a))$$

Let us consider the level sets of V : $6s - 9x = c$, $s = \frac{3}{2}x + c$

Any other solution with

$$(a, u(a)) \in \text{Line 1}$$

$$(b, u(b)) \in \text{Line 2}$$

satisfies the same equality,
and therefore $u_0(x) = 3x$
minimizes

$$F(u) = \int_a^b \dot{u}^2$$

among all functions u with endpoints in line 1 and line 2.

In this way we actually have proved minimality for a curve to curve problem.

Moreover, we have equality iff $\dot{u} = 3$, and therefore all functions

$$u(x) = 3x + k$$

are minimum points as well.

Notation

$V(x, s)$ is usually
called "verification function"

