

**Prop.** Let us consider a quadratic functional.

Then there exists a constant  $M \in \mathbb{R}$  such that

$$|Q(v)| \leq M \int_a^b \dot{v}^2 dx \quad \forall v \text{ admissible } (v(a)=v(b)=0)$$

The constant  $M$  depends on  $\|A\|_\infty$ ,  $\|B\|_\infty$ ,  $\|C\|_\infty$  and  $b-a$ .

**Proof**  $|Q(v)| \leq \int_a^b |A| v^2 + 2 \int_a^b |B| |v \dot{v}| + \underbrace{\int_a^b |C| \dot{v}^2}_{\leq \|C\|_\infty \int_a^b \dot{v}^2}$

Since  $v(a)=v(b)=0$  we can always estimate

$\|v(x)\|_{L^\infty}$  in terms of  $\left(\int_a^b \dot{v}^2\right)^{1/2}$  ... conclusion is standard.

Remark For every  $\delta > 0 \exists \varepsilon > 0$  such that

$$\|A\|_\infty \leq \varepsilon, \quad \|B\|_\infty \leq \varepsilon, \quad \|C\|_\infty \leq \varepsilon \Rightarrow M \leq \delta$$

**Prop 2** Assume that  $(L^+)$  and  $(J^+)$  are satisfied.

Then there exists  $\varepsilon > 0$  such that

$$Q(v) \geq \varepsilon \int_a^b \dot{v}^2 dx \quad \forall \text{ admissible } v$$

**Proof**  $\hat{Q}(v) := \int_a^b A v^2 + 2 B v \dot{v} + (C - \varepsilon) \dot{v}^2$

Claim: if  $\varepsilon$  is small enough, then  $\hat{Q}$  still satisfies  $(L^+)$  and  $(J^+)$

$$Q \text{ satisfies } (L^+) \Rightarrow \inf_{x \in [a,b]} C(x) > 0 \Rightarrow \inf_{x \in [a,b]} (C(x) - \varepsilon) > 0 \text{ when } \varepsilon \text{ is small}$$

In order to prove (J<sup>+</sup>) we have to solve (JDE) for  $\hat{Q}$ .  
This equation is of the form

$$\ddot{v} = \alpha_\varepsilon(x) \dot{v} + \beta_\varepsilon(x) v \quad (\text{JDE}_\varepsilon)$$

where  $\alpha_\varepsilon(x) \rightarrow \alpha(x)$  and  $\beta_\varepsilon(x) \rightarrow \beta(x)$  UNIFORMLY as  $\varepsilon \rightarrow 0$ .

We know that the limit eqn

$$\ddot{v} = \alpha(x) \dot{v} + \beta(x) v$$

$$v(a) = 0$$

$$\dot{v}(a) = 1$$

has a solution  $v(x)$  which does not vanish in  $(a, b]$ .

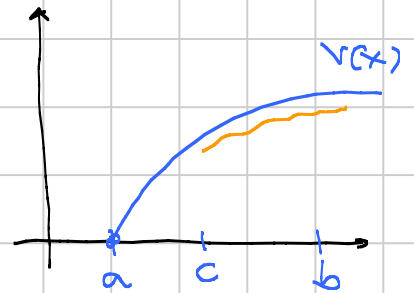
Claim: (JDE<sub>ε</sub>) with initial data  $v_\varepsilon(a) = 0$ ,  $\dot{v}_\varepsilon(a) = 1$   
has the solution which does not vanish in  $(a, b]$ .

From the theory of ODEs we know that  $v_\varepsilon \rightarrow v$  and  $\dot{v}_\varepsilon \rightarrow \dot{v}$  unif. in  $[a, b]$ .

There exists  $c \in (a, b)$  s.t.

$$\dot{v}(x) \geq \frac{1}{2} \quad \forall x \in [a, c]$$

$$v(x) \geq m > 0 \quad \forall x \in [c, b]$$



Due to unif. conv.,  $v_\varepsilon(x) \geq \frac{m}{2} \quad \forall x \in [c, b]$  for  $\varepsilon$  small

and  $\dot{v}_\varepsilon(x) \geq \frac{1}{4} \quad \forall x \in [a, c]$  "

This implies that  $v_\varepsilon$  is strictly increasing in  $[a, c]$  for  $\varepsilon$  small, and hence it cannot vanish in  $(a, c]$ .

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**Prop 3** Assume that  $Q(v)$  satisfies  $(L^+)$  and  $(J^+)$   
 Let us consider a new quadratic functional  $\hat{Q}(v)$   
 with coeff.  $\hat{A}, \hat{B}, \hat{C}$  s.t.

$$\|\hat{A} - A\|_\infty \leq \varepsilon, \quad \|\hat{B} - B\|_\infty \leq \varepsilon, \quad \|\hat{C} - C\|_\infty \leq \varepsilon$$

If  $\varepsilon > 0$  is small enough, then  $\hat{Q}(v) \geq 0$  for every admissible  $v$ .

**Proof.** Let  $\hat{\hat{Q}}(v) := Q(v) - \hat{Q}(v)$ , so that

$$\hat{Q}(v) = Q(v) - \hat{\hat{Q}}(v) \geq Q(v) - |\hat{\hat{Q}}(v)|$$

Use that  $Q(v) \geq \varepsilon_0 \int_a^b \dot{v}^2$  (Prop. 2)

$$|\hat{\hat{Q}}(v)| \leq M \int_a^b \dot{v}^2 \quad (\text{Prop. 4})$$

If  $\varepsilon$  is small enough, then  $M$  is as small as we need,  
 and in particular  $M \leq \varepsilon_0$ .

This concludes the proof.

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**Back to second variation** Let

$$F(u) := \int_a^b L(x, u(x), \dot{u}(x))$$

$$\delta F(u, v) := \int_a^b \underbrace{L_{ss}(x, u, \dot{u})}_{A(x)} v^2 + 2 \underbrace{L_{sp}(\cdot)}_{B(x)} v \dot{v} + \underbrace{L_{pp}(\cdot)}_{C(x)} \dot{v}^2$$

**Necessary conditions**

$$\text{no DLM} \Rightarrow (E) + (L) + [(L^+) \Rightarrow (J)]$$

with DBC  $\uparrow$   
Euler-Lagr  
equ

## Sufficient conditions

$$(E) + (L^+) + (J^+) \Rightarrow \text{no WLM}$$

Proof. We need to prove that

$$F(u_0 + v) \geq F(u_0)$$

provided that the  $C^1$ -norm of  $v$  is small. Let us introduce

$$\psi(t) := F(u_0 + tv)$$

$$\text{Then } F(u_0 + v) = \psi(1) = \overset{0(E)}{\psi(0)} + \overset{\text{Taylor}}{\psi'(0)} \cdot 1 + \frac{1}{2} \psi''(t_0)$$

$$\text{For some } t_0 \in (0, 1) \quad = F(u_0) + \frac{1}{2} \psi''(t_0)$$

So it is enough to prove that  $\psi''(t_0) \geq 0$ . But

$$\begin{aligned} \psi''(t_0) &= \int_a^b L_{ss}(x, u_0(x) + t_0 v(x), \dot{u}_0(x) + t_0 \dot{v}(x)) v^2(x) + \text{other 2 terms} \\ &= Q^{(t_0)}(v) \end{aligned}$$

Idea: we know that  $Q^{(0)}(v)$  satisfies  $(J^+)$  and  $(L^+)$ , and therefore it is strictly positive.

If  $L$  is of class  $C^3$  (but maybe  $C^2$  is enough) and  $\|v\|_\infty$  and  $\|\dot{v}\|_\infty$  are small, then the coeff. of

$Q^{(t_0)}$  are as close as needed to the coeff. of  $Q^{(0)}$ .

We conclude by exploiting prop. 3.

THE Example Consider

$$F(u) = \int_0^l \dot{u}^2 - u^2$$

$(L^+)$  is trivial  $C(x) \equiv 1$ . (JDE)  $\ddot{u} = -u$

Need to solve with  $u(0) = 0, \dot{u}(0) = 1$

The solution is  $u(x) = \sin x$ . The first cong. point is  $\pi$ .

Conclusion:

- 1 - if  $l > \pi$ , then (J) not satisfied, hence  $F(u)$  is not nonnegative
- 2 - if  $l < \pi$ , then  $(L^+)$  and  $(J^+)$  are satisfied, and therefore  $F(u) \geq 0$
- 3 - if  $l = \pi$ , then  $F(u) \geq 0$  again.

Proof. If  $F$  is not nonnegative, then the infimum is  $-\infty$ .  
Assume that  $F_\pi$  has  $\inf = -\infty$ .  
This means that

$$F_\pi(u) \leq -2016$$

for some  $u$  vanishing at the boundary

Rescale  $u$  to the interval  $[0, \pi - \varepsilon]$   
with an affine transformation

$$\text{Then } \int_0^{\pi-\varepsilon} \dot{u}_\varepsilon^2 - u_\varepsilon^2 \rightarrow -2016$$

and therefore  $F_{\pi-\varepsilon}(u_\varepsilon) < 0$  for  $\varepsilon$  small.  
This contradicts that

$F_{\pi-\varepsilon}$  is nonnegative  
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