

$$Q(v) = \int_a^b A(x) v^2 + 2B(x) v \dot{v} + C(x) \dot{v}^2$$

$$(L) \quad C(x) \geq 0 \quad \forall x \in [a, b]$$

$$(L^+) \quad C(x) > 0 \quad \forall x \in [a, b]$$

$$(JDE) \quad [C(x) \dot{v} + B(x) v]' = A(x) v + B(x) \dot{v}$$

$\leadsto$ Conjugate points (wrt a or b)

(J) No conj. point in (a, b)

(J⁺) No conj. point of a in $[a, b]$

Necessary conditions

$$Q(v) \geq 0 \quad \forall v \in C_c^\infty(a, b) \quad \Rightarrow \quad (L) + [(L^+) \Rightarrow (J)]$$

(L⁺) $\Rightarrow$ can write (JDE) in normal form $\Rightarrow$ ex./uniq./reg.

Sufficient conditions

$$(L^+) + (J^+) \Rightarrow Q(v) \geq 0 \quad \forall v \in C_c^\infty(a, b)$$

ROAD MAP

$$\textcircled{1} \quad (L^+) + (J^+) \Rightarrow (J^{++})$$

$$\textcircled{2} \quad (J^{++}) \Rightarrow (R)$$

$$\textcircled{3} \quad (R) \Rightarrow Q(v) \text{ is nonnegative}$$

(J⁺⁺) There exists a solution w to (JDE) such that
 $w(x) > 0 \quad \forall x \in [a, b]$

(R) There exists $z: [a, b] \rightarrow \mathbb{R}$ solution to

$$z' = \frac{[z + B(x)]^2}{C(x)} - A(x) \quad (\text{RDE})$$

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RICCATI

Remark 1 - (L⁺) allows to divide by $C(x)$

2 - The RHS of (RDE) is quadratic in z : no guarantee that solutions starting from a survive up to b .

Proof of ① Due to (J⁺) we know that (JDE) has a solution $w_1: [a, b] \rightarrow \mathbb{R}$ s.t.

$$w_1(x) > 0 \quad \forall x \in [a, b]$$

In a similar way (oscillation lemma) we know that (JDE) has a solution w_2 with

$$w_2(x) > 0 \quad \forall x \in [a, b]$$

Since (JDE) is linear, $w_1 + w_2$ is again a solution and it is strictly positive in $[a, b]$.

Proof of ② Let us take w as in step ①. Let us set

$$z(x) := - \frac{[C(x)\dot{w}(x) + B(x)w(x)]}{w(x)} \leftarrow \text{LHS of (JDE)}$$

$$= - C(x) \frac{\dot{w}(x)}{w(x)} - B(x)$$

Claim: z solves (RDE).

$$\text{RHS of (RDE)}: \frac{[z + B]^2}{C} - A = C \frac{\dot{w}^2}{w^2} - A$$

$$\text{LHS of (RDE)}: z' = - \frac{[C\dot{w} + Bw]'w - \dot{w}[C\dot{w} + Bw]}{w^2}$$

$$\text{use (JDE)} \leftarrow = - \frac{[B\dot{w} + A w]w - \dot{w}[C\dot{w} + Bw]}{w^2} = -A + C \frac{\dot{w}^2}{w^2}$$

VERY SUFFICIENT CONDITION

and assume that

Assume $C(x) \geq 0$ in $[a, b]$

$$A(x) \cdot C(x) \geq B(x)^2 \quad \forall x \in [a, b]$$

Then $Q(v) \geq 0$ for every $v \in H^1([a, b])$ indep. of B .

Proof. The condition guarantees that the matrix

$$\begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix}$$

is a nonnegative matrix (i.e. nonnegative eigenvalues) and therefore

$$A(x) v^2(x) + 2B(x) v(x) \dot{v}(x) + C(x) \dot{v}^2(x) \geq 0$$

for every $x \in [a, b]$.

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Proof of ③ Let $z(x)$ be a solution to (RDE) defined in $[a, b]$

Let us consider

$$\dot{z}(x) v^2(x) + 2z(x) v(x) \dot{v}(x)$$

$$\hat{Q}(v) := Q(v) + \int_a^b [z(x) v^2(x)]'$$

$$= \int_a^b [A(x) + \dot{z}(x)] v^2 + 2[B(x) + z(x)] v \dot{v} + C(x) \dot{v}^2$$

We observe that $\hat{Q}(v) = Q(v)$ for every admissible v .

This is because

$$\int_a^b [z(x) v^2(x)]' dx = [z(x) v^2(x)]_a^b = 0 \quad \text{because of BC on } v.$$

Surprise: $\tilde{Q}$ satisfies the "very sufficient condition":

$$[A(x) + \tilde{z}(x)] \cdot C(x) \geq [B(x) + z(x)]^2$$

Asking = is just (RDE)

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Comment 1 (JDE) vs (RDE)

Riccati equ $z' = A(x)z^2 + B(x)z + C(x)$

Second order linear ODE : $v'' = \alpha(x)v' + \beta(x)v$

Assume we want a solution to 2nd order ODE which is nonnegative. Reasonable: $v = e^w$. Let us write the equ. for w :

$$\dot{v} = \dot{w} e^w, \quad \ddot{v} = \ddot{w} e^w + \dot{w}^2 e^w \quad \text{so that}$$

$$\cancel{\ddot{w} e^w} + \cancel{\dot{w}^2 e^w} = \alpha(x) \cancel{\dot{w} e^w} + \beta(x) \cancel{e^w}$$

$$z = \dot{w} \text{ solves } z' + z^2 = \alpha(x)z + \beta(x)$$

which is a Riccati equ. Moreover $v = e^w$ implies $w = \log v$ and therefore

$$z = \dot{w} = \frac{\dot{v}}{v} \rightsquigarrow \text{this reminds us of the substitution we did before}$$

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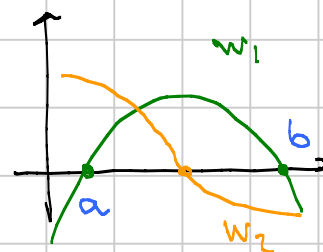
Oscillation Lemma Assume w_1 and w_2 are two lin. indep. solutions to

$$\ddot{v} = \alpha(x)\dot{v} + \beta(x)v.$$

Then between any two consecutive zeroes of $w_1(x)$ there is exactly one zero of w_2

Proof. Let us consider

$$W(x) := \det \begin{pmatrix} w_1(x) & \dot{w}_1(x) \\ w_2(x) & \dot{w}_2(x) \end{pmatrix}$$
$$= w_1 \dot{w}_2 - \dot{w}_1 w_2$$



$$\begin{aligned} \dot{W} &= \cancel{\dot{w}_1} \dot{w}_2 + w_1 \ddot{w}_2 - \ddot{w}_1 w_2 - \cancel{\dot{w}_1} \dot{w}_2 \\ &= w_1 (\alpha(x) \dot{w}_2 + \beta(x) w_2) - w_2 (\alpha(x) \dot{w}_1 + \beta(x) w_1) \\ &= \alpha(x) W \end{aligned}$$

So W solves $\dot{W} = \alpha(x) W \Rightarrow$ the sign of $W(x)$ is constant

$$W(a) = -\dot{w}_1(a) w_2(a) < 0 \Rightarrow W(b) < 0$$

$$W(b) = -\underbrace{\dot{w}_1(b)}_{-} \underbrace{w_2(b)}_{+} < 0 \Rightarrow w_2(b) < 0$$

$\Rightarrow w_2$ changes its sign in $[a, b]$. Why only once?

If not, we swap w_1 and w_2 and we obtain one more 0 of w_1 .

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