

## QUADRATIC FUNCTIONALS

Def. A quadr. functional is the following

$$Q(v) := \int_a^b A(x) v^2 + 2B(x) v \dot{v} + C(x) \dot{v}^2$$

where  $A(x)$ ,  $B(x)$  and  $C(x)$  are given functions defined in  $[a, b]$ .

Remark We consider different assumptions on  $A, B, C$

- $A, B, C$  in  $L^\infty([a, b])$  is enough in order to define  $Q(v)$
- $A, B, C$  in  $C^0$
- $A, B, C$  in  $C^1$

Def. A quadratic functional is nonnegative (positive semi-def.) if

$$Q(v) \geq 0 \quad \forall v \in C_c^\infty([a, b])$$

Prop. The following are equivalent

- (i)  $Q(v) \geq 0$  for every  $v \in C_c^\infty([a, b])$ ,
- (ii)  $Q(v) \geq 0$  " "  $v \in C^1([a, b])$  with  $v(a) = v(b) = 0$ ,
- (iii)  $Q(v) \geq 0$  " "  $v \in C^1([a, b])$  " "
- (iv)  $Q(v) \geq 0$  " "  $v \in H_0^1([a, b])$  " "

Proof Standard approximation argument.

Goal of the theory: discovering necessary/suff. conditions on  $A(x)$ ,  $B(x)$ ,  $C(x)$  that guarantee that  $Q(v)$  is nonnegative.

# LEGENDRE CONDITION

$$(L) \quad C(x) \geq 0 \quad \forall x \in [a, b]$$

$$(L^+) \quad C(x) > 0 \quad \forall x \in [a, b]$$

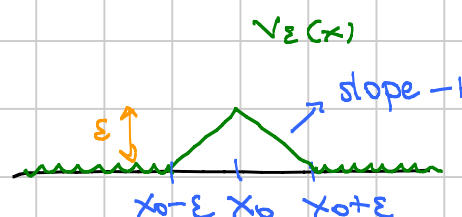
**Theorem 1** Assume that  $A, B, C$  are continuous. Then

$$Q(v) \text{ nonnegative} \Rightarrow (L)$$

**Proof.** We know that  $Q(v) \geq 0$  for every  $v \in PW C^1$ .  
Take any  $x_0 \in (a, b)$  and for  $\varepsilon > 0$   
define  $v_\varepsilon$  as in the figure

$$0 \leq Q(v_\varepsilon)$$

$$\begin{aligned} &= \int_{x_0-\varepsilon}^{x_0} A(x) v_\varepsilon^2 + 2B(x) v_\varepsilon(x) + C(x) \\ &+ \int_{x_0}^{x_0+\varepsilon} A(x) v_\varepsilon^2 - 2B(x) v_\varepsilon(x) + C(x) \end{aligned}$$



Note that  $v_\varepsilon$  is admissible if  $\varepsilon$  is small enough.  
Divide by  $\varepsilon$  and apply the mean value theorem for integrals

$$\begin{aligned} 0 \leq & A(x_\varepsilon) v_\varepsilon^2(x_\varepsilon) + 2B(x_\varepsilon) v_\varepsilon(x_\varepsilon) + C(x_\varepsilon) \\ & + A(y_\varepsilon) v_\varepsilon^2(y_\varepsilon) - 2B(y_\varepsilon) v_\varepsilon(y_\varepsilon) + C(y_\varepsilon) \end{aligned}$$

Therefore  $C(x_0) \geq 0$ .



Example

$$Q(v) = \int_0^l \sin x \cdot v^2 + \cos x \cdot \dot{v}^2$$

This cannot be nonnegative if  $l \geq \frac{\pi}{2}$

## JACOBI CONDITION and CONJUGATE POINTS

Let us assume that  $A, B, C$  are of class  $C^1$ . Consider the ODE

$$(C\dot{v} + Bv)' = B\dot{v} + Av \quad (\text{JDE})$$

This is called JACOBI DIFF. EQN

Note: this is exactly Euler-Lagrange eqn for  $\mathcal{Q}(v)$

$$L(x, s, p) : A(x) s^2 + 2B(x) sp + C(x) p^2$$

Remark If  $A, B, C$  and  $v$  are regular enough we can write JDE as

$$C'\dot{v} + C\ddot{v} + B'\dot{v} + \cancel{B\dot{v}} = \cancel{B\dot{v}} + Av$$

Under assumption  $(L^+)$  we can write it in normal form

$$\ddot{v} = \frac{C'}{C} \dot{v} + \frac{A - B'}{C} v$$

This is a second order linear eqn of the form

$$\ddot{v} = \alpha(x)\dot{v} + \beta(x)v$$

for which the usual theory (existence, uniqueness, gl. ex.) applies.

Def. We say that a point  $c$  is conjugate to  $a$  if (JDE) has a solution such that

$$v(a) = 0, \quad \dot{v}(a) = 1, \quad v(c) = 0$$

In other words, solve (JDE) with initial data  $v(a)=0, \dot{v}(a)=1$ .  
All other 0's of  $v$  are conjugate points of  $a$ .

Remark Solving with  $v(a)=0, \dot{v}(a)=k \neq 0$  produces a multiple of the previous solution, and therefore the same cony. points.

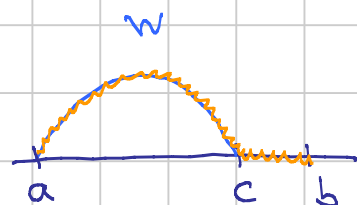
**JACOBI CONDITIONS** (J) No cony. points of  $a$  in  $(a,b)$ .  
(J<sup>+</sup>) " " " " " "  $(a,b]$

Theorem Assume  $A, B, C$  of class  $C^1$ .  
Assume that  $(L^+)$  is satisfied  
Assume that  $Q(v)$  is nonnegative.  
Then (J) is satisfied

Proof. Assume by contrad. that a cony. point exists in  $(a,b)$ .  
Let  $w$  be the con. solution of (JDE) with  
 $w(a)=0 \quad \dot{w}(a)=1 \quad w(c)=0$

Define

$$v(x) := \begin{cases} w(x) & \text{if } x \in [a, c] \\ 0 & \text{if } x \in [c, b] \end{cases}$$



Claim:  $Q(v) = 0$ . Suppose this is true.

Then  $v$  is a GM for  $Q$ , and hence it is a solution to Euler-lagr eqn of  $Q$  which is JDE. But  $v(x)$  is not of class  $C^1$  in  $x=c$ , but all solutions to JDE are of class  $C^2$  in  $[a,b]$ .

(note that  $\dot{w}(c) \neq 0$  because otherwise  $w(x) \equiv 0$ ).

In order to prove the claim it is enough to prove that

$$\int_a^c A w^2 + 2B w \dot{w} + C \dot{w}^2 = 0$$

This follows from JDE after integration by parts

$$(C\dot{w} + Bw)' \cdot w = (B\dot{w} + Aw)w$$

$$\int_a^c (C\dot{w} + Bw)' w = \int_a^c Bw\dot{w} + Aw^2$$

$$- \int C\dot{w}^2 + Bw\dot{w} + \text{boundary term which is 0 because } w=0 \text{ in } a \text{ and } c.$$

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Remark We could define conj. points relative to  $b$  by solving JDE with "initial" data

$$w(b) = 0 \quad \dot{w}(b) = \pm 1$$

Fact. There exists a point  $c \in (a, b)$  conj. to  $a \iff$   
" " " " " " to  $b$

(under assumption  $(L^+)$ )

Lemma (oscillation lemma) Take any two lin. indep. solutions  $w_1$  and  $w_2$  to a lin. equ.

$$\ddot{v} = \alpha(x)\dot{v} + \beta(x)v$$

Then between any two zeroes of  $w_1$  there exists at least one zero of  $w_2$

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