

Four notions of minimum point

Setting: integral functionals

$$F(u) := \int_a^b L(x, u, \dot{u}) dx$$

$$\text{in } X = \{u \in C^1([a, b]) : u(a) = A, u(b) = B\}$$

Def. we say that  $u_0 : [a, b] \rightarrow \mathbb{R}$  is

- **GLOBAL MINIMUM** (GM) if  $F(u) \geq F(u_0)$  for every  $u \in X$ ,
- **STRONG LOCAL MINIMUM** (SLM) if  $u_0$  is a loc. min. with respect to the  $C^0$  metric, namely

$$\exists \delta > 0 \text{ s.t. } F(u) \geq F(u_0) \quad \forall u \in X \text{ with } \|u - u_0\|_{C^0} \leq \delta$$

$\sup_{x \in [a, b]} |u(x) - u_0(x)|$

- **WEAK LOCAL MINIMUM** (WLM) ...  $C^1$  metric ...

$$\exists \delta > 0 \text{ s.t. } F(u) \geq F(u_0) \quad \forall u \in X \text{ with } \|u - u_0\|_{C^0} \leq \delta$$

$\|\dot{u} - \dot{u}_0\|_{C^0} \leq \delta$

- **DIRECTIONAL LOCAL MINIMUM** (DLM) if the function

$$\psi(t) := F(u_0 + t v) \quad \psi : (-\delta, \delta) \rightarrow \mathbb{R}$$

has a local minimum for  $t = 0$ .

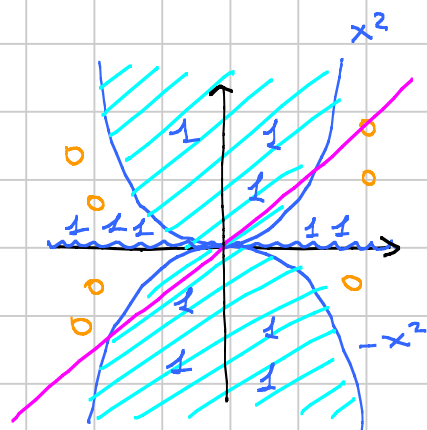
Remark It is easy to see that

$$GM \Rightarrow SLM \Rightarrow WLM \Rightarrow DLM$$

All other implications are false

Example 1 Consider  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x,y) := \begin{cases} 1 & (x,y) \in \text{shaded region} \\ & \text{(including } y=0) \\ 0 & \text{otherwise} \end{cases}$$



The origin is a DLM but not a local min. in any sense.

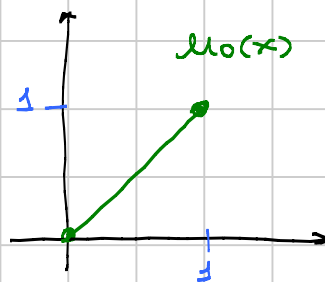
Example 2  $F(u) = \int_0^1 \dot{u}^3 dx$   $u(0)=0, u(1)=1$

We claim that  $u_0(x) := x$  is a

SLM but not a WLM.

WLM

SLM (admitted after video)



SLM Claim:  $u_0(x)$  is the minimum point in

WLM (after video)

$$\mathcal{U} = \{u \in C^1([0,1]) : \dot{u}(x) \geq 0, u(0)=0, u(1)=1\}$$

Proof. Let us consider

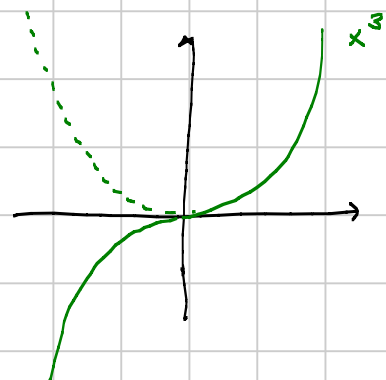
$$G(u) := \int_0^1 |\dot{u}|^3 dx$$

$u_0(x)$  is a minimum point for  $G(u)$

subject to the BC. It is also the

unique GM because the Lagrangian is strictly convex.

On the other hand  $F(u) = G(u)$  in  $\mathcal{U}$ , and therefore



$$F(u) = G(u) \geq G(u_0) = F(u_0) \quad \forall u \in U.$$

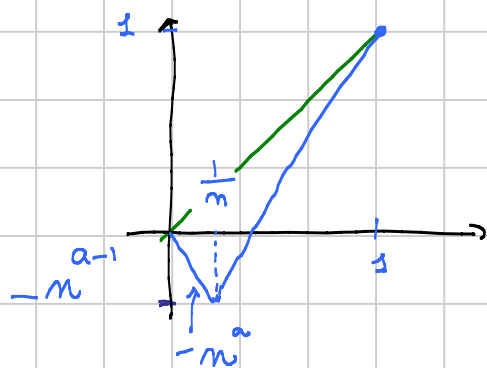
WLM

SLM

SLM ← after video

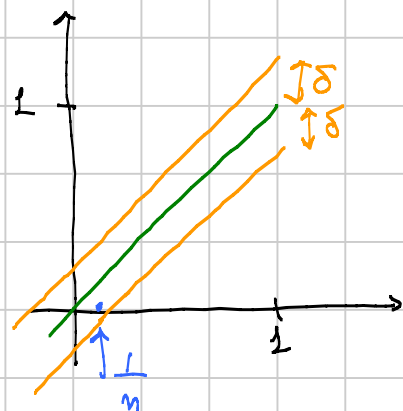
We claim that  $u_0(x) = x$  is NOT a WLM, namely in any  $C^0$ -neighborhood of  $u_0$  there exists  $u(x)$  with  $F(u) < F(u_0)$

Proof. Consider  $u$  like below



If  $a < 1$ , then  $u_n(x)$  is as close as we want to  $u_0(x)$

$$\begin{aligned} F(u_n) &= \int_0^{1/n} (-n^a)^3 + \int_{1/n}^1 u_n^3 \\ &= -n^{3a-1} + \int_{1/n}^1 u_n^3 \\ &\quad \underbrace{\hspace{10em}}_{\substack{\rightarrow -\infty \\ \text{if } a > \frac{1}{3}}} \quad \underbrace{\hspace{10em}}_{\substack{\rightarrow 1 \\ \text{if } a < 1}} \end{aligned}$$



Conclusion: fix any  $a \in (\frac{1}{3}, 1)$ . Then  $u_n \rightarrow u_0$  in the  $C^0$ -norm, but

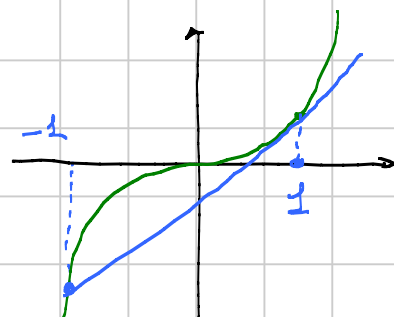
$$F(u_n) \rightarrow -\infty$$

Remark ① It is not difficult to construct a sequence of  $C^2$  functions converging to  $u_0$  in  $C^0$ .

② The sequence  $u_n$  does not belong eventually to any  $C^1$ -ubd of  $u_0$ .

③ We can prove that  $u_0$  is a SLM wrt a  $C^1$ -ubd larger than  $U$ . Claim:  $u_0$  is wlm in

$$U_1 := \{u \in C^1 : u(x) \geq -1 + B.c.\}$$

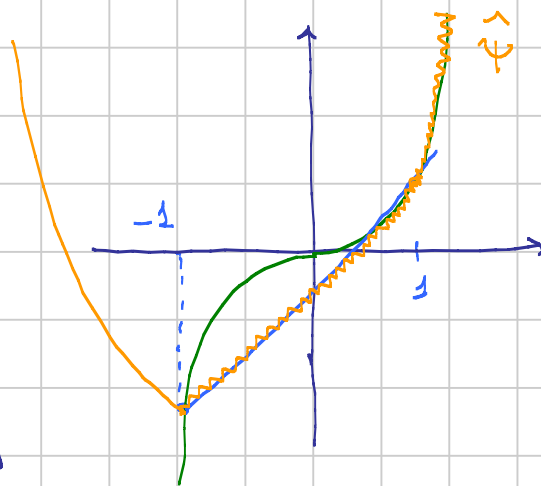


Proof. The same as before with

$$\hat{G}(u) := \int_0^1 \hat{\psi}(u) du$$

where  $\hat{\psi}$  is any convex function s.t.

$$\hat{\psi}(x) = \begin{cases} x^3 & x \geq 1 \\ \text{tang. line} & x \in [-1, 1] \\ \dots & x \leq -1 \end{cases}$$



One again  $F(u) \geq \hat{G}(u) \geq \hat{G}(u_0) = F(u_0)$

$\uparrow$   $\uparrow$   $\uparrow$   
 $\forall u \in U_1$  convexity  $\hat{\psi}(1) = 1$   
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## SECOND VARIATION OF A FUNCTIONAL

$F(u)$  any functional,  $v$  be any admissible direction

$$\psi(t) := F(u + tv) \quad t \in (-\delta, \delta)$$

The second variation of  $F$  in the point  $u$  wrt the dir  $v$  is

$$\delta^2 F(u, v) := \psi''(0)$$

If  $F(u) = \int_a^b L(x, u, \dot{u}) dx$  with DBC, then

$$F(u + tv) = \int_a^b L(x, \underbrace{u + tv}_S, \underbrace{\dot{u} + t\dot{v}}_P) dx$$

$$\psi'(t) = \int_a^b L_S(x, u + tv, \dot{u} + t\dot{v}) v + L_P(x, u + tv, \dot{u} + t\dot{v}) \dot{v}$$

$$\psi''(t) = \int_a^b L_{SS}(x, u + tv, \dot{u} + t\dot{v}) v^2 + 2 L_{SP}(\dots) v \dot{v} + L_{PP}(\dots) \dot{v}^2$$

In conclusion, for the standard functional  $L(x, s, p)$

$$\delta^2 F(u, v) = \int_a^b L_{ss}(x, u, \dot{u}) v^2 + 2 L_{sp}(\dots) v \dot{v} + L_{pp}(\dots) \dot{v}^2$$

Prop. Assume that  $L(x, s, p)$  is of class  $C^2$ .  
Assume that  $u_0$  is a DLM for  $F(u)$ .

Then

- $u_0(x)$  is a solution to the Euler-Lagrange equ

$$\psi'(0) = \delta F(u_0, v) = 0 \quad \forall v \text{ admissible}$$

- $u_0(x)$  satisfies

$$\psi''(0) = \delta^2 F(u_0, v) \geq 0 \quad \forall v \text{ admissible}$$

Remark When  $u_0$  is fixed, the second variation is a quadratic functional in  $v$ .

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