

LAGRANGE MULTIPLIERS in $\mathbb{R}^n$

Setting: ball in $\mathbb{R}^n$: $B_r(x_0)$ $x_0 \in \mathbb{R}^n$ $r > 0$
 $f: B_r(x_0) \rightarrow \mathbb{R}$ (closed ball)

$\Phi_1, \dots, \Phi_k: B_r(x_0) \rightarrow \mathbb{R}$ s.t. $\Phi_1(x_0) = \dots = \Phi_k(x_0) = 0$
 Define

$$V := \{x \in B_r(x_0) : \underbrace{\Phi_1(x) = \dots = \Phi_k(x) = 0}_{\text{eqns of } V}\}$$

Theorem Assume that $f, \Phi_1, \dots, \Phi_k$ are of class C^1 in $B_r(x_0)$.
 Assume

$$x_0 \in \argmin \{f(x) : x \in V\}$$

Then

- either $\nabla \Phi_1(x_0), \dots, \nabla \Phi_k(x_0)$ are NOT linearly indep.
- or $\nabla f(x_0) \in \text{Span}\{\nabla \Phi_1(x_0), \dots, \nabla \Phi_k(x_0)\}$, namely there exist $\lambda_1, \dots, \lambda_k$ such that

$$\nabla f(x_0) = \lambda_1 \nabla \Phi_1(x_0) + \dots + \lambda_k \nabla \Phi_k(x_0)$$

Proof. Step 1 wlog we can assume that x_0 is a strict local minimum, namely

$$f(x) > f(x_0) \quad \forall x \in V \setminus \{x_0\}$$

If not, replace $f(x)$ by $\hat{f}(x) := f(x) + \|x - x_0\|^2$. Now $\hat{f}$ has a strict minimum in x_0 and

$$\nabla f(x_0) = \nabla \hat{f}(x_0).$$

Step 2 Let us consider the new function

$$F_n(x) := f(x) + n\Phi_1^2(x) + \dots + n\Phi_k^2(x) \quad x \in B_n(x_0)$$

Claim: for every $n \geq 1$, the function F_n has a minimum in $B_n(x_0)$ (trivial: F_n is cont. and $B_n(x_0)$ is compact) and

$$\min\{F_n(x) : x \in B_n(x_0)\} \leq f(x_0) \quad (*)$$

(this is because $\min \leq F_n(x_0) = f(x_0)$)

Step 3 For every $n \in \mathbb{N}$, let x_n denote a min. point of F_n .
Up to subsequences (not relabeled) $x_n \rightarrow x_\infty$
Claim: $x_\infty = x_0$.

Let us prove that $x_\infty \in V$. If not $\exists i \in \{1, \dots, k\}$ s.t. $\Phi_i(x_\infty) \neq 0$ and therefore

$$\liminf_{n \rightarrow \infty} F_n(x_n) \geq \liminf_{n \rightarrow \infty} \underbrace{f(x_n)}_{\downarrow f(x_\infty)} + n \underbrace{\Phi_i^2(x_n)}_{\substack{\uparrow \Phi_i^2(x_\infty) \\ \downarrow +\infty}}$$

and this contradicts (*)

Let us prove that actually $x_\infty = x_0$.

$$\liminf F_n(x_n) \geq \liminf f(x_n) = f(x_\infty)$$

and once again this contradicts (*)

Step 4 Since $x_n \rightarrow x_0$, then eventually $x_n \in \text{Int}(B_n(x_0))$ and hence

$$\nabla F_n(x_n) = 0 \quad \text{for } n \text{ large}$$

But

$$\nabla F_m(x_m) = \nabla f(x_m) + 2m\phi_1(x_m)\nabla\phi_1(x_m) + \dots + 2m\phi_k(x_m)\nabla\phi_k(x_m) = 0$$

This says that $\nabla f(x_m) \in \text{Span}\{\nabla\phi_1(x_m), \dots, \nabla\phi_k(x_m)\}$

$\downarrow$ $\downarrow$ $\downarrow$

$\nabla f(x_0)$ $\nabla\phi_1(x_0), \dots, \nabla\phi_k(x_0)$

General fact about linear algebra:

→ either $\nabla\phi_1(x_0), \dots, \nabla\phi_k(x_0)$ are linearly dependent

→ or $\nabla f(x_0) \in \text{Span}\{\dots\}$

Proof Let us consider the matrix

$$M_m = (\nabla f(x_m) \mid \nabla\phi_1(x_m) \mid \dots \mid \nabla\phi_k(x_m))$$

By assumption $\text{rank}(M_m) \leq k \quad \forall m \in \mathbb{N}$

Now $M_m \rightarrow M_\infty$ and we need that $\text{rank}(M_\infty) \leq k$.

Main point: the set of matrices with $\text{rank} \leq k$ is closed

$\text{rank} \leq k \Leftrightarrow$ all minors $(k+1) \times (k+1)$ are $= 0$
and this is a closed relation.

Remark Equivalent to saying that the rank is a LSC function.

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Comment Penalization method: all points compete for the min, but points outside V are penalized.

Interpretation by means of Γ -conv.

$$\Gamma\text{-}\lim_{n \rightarrow \infty} F_n(x) = F_\infty(x) := \begin{cases} f(x) & \text{if } x \in V \\ +\infty & \text{otherwise} \end{cases}$$

What we have proved is the convergence of minima and minimizers, because x_0 is the unique element in $\arg\min F_\infty$

Example

$$M_n = \min \left\{ n \int_0^1 \dot{u}^2 + \int_0^1 \arctan u : u \in C^1, u(0)=0, u(1)=7 \right\}$$

Fact 1 M_n is well defined for every $n \in \mathbb{N}$

(Standard application of direct method)

Fact 2 $\lim_{n \rightarrow \infty} M_n = +\infty$

$$F_n(u) := \int_0^1 \dot{u}^2 + \frac{1}{n} \int_0^1 \arctan u$$

$$M_n = n \cdot \min \{ F_n(u) : u(0)=0, u(1)=7 \}$$

Assume that $F_n \xrightarrow{\Gamma} F_\infty$ with equicoerc. Then

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = \min \{ F_\infty(u) \}$$

In this case the Γ -lim is also the unif. lim. (for ex in C^0)

$$F_\infty(u) = \int_0^1 \dot{u}^2 \quad \text{if } u \text{ in } H^1 \text{ satisfies BC}$$

$$\min F_\infty = 49 \quad \Rightarrow \quad M_n = 49n + o(n)$$