

(X, d) metric space, $F_n: X \rightarrow \bar{\mathbb{R}}$

$$\Gamma\text{-}\liminf_{n \rightarrow \infty} F_n(x) := \inf_{n \rightarrow \infty} \{ \liminf_{m \rightarrow \infty} F_n(x_m) : x_m \rightarrow x \}$$

$$\Gamma\text{-}\limsup_{n \rightarrow \infty} F_n(x) := \inf_{n \rightarrow \infty} \{ \limsup_{m \rightarrow \infty} F_n(x_m) : x_m \rightarrow x \}$$

If $\Gamma\text{-}\liminf F_n(x) = \Gamma\text{-}\limsup F_n(x)$ for every x , the common value is the Γ -limit.

- Goals:
- ① Γ -liminf and Γ -limsup are LSC in X
 - ② Convergence of minima and minimizers

Lemma 1 (Gamma convergence and subsequences)

Let n_k be an increasing sequence of integers.

Then

$$\begin{aligned} \Gamma\text{-}\liminf_{n \rightarrow \infty} F_n(x) &\leq \Gamma\text{-}\liminf_{k \rightarrow \infty} F_{n_k}(x) \leq \Gamma\text{-}\limsup_{k \rightarrow \infty} F_{n_k}(x) \\ &\leq \Gamma\text{-}\limsup_{n \rightarrow \infty} F_n(x) \end{aligned}$$

In particular, if $\Gamma\text{-}\lim_{n \rightarrow \infty} F_n(x)$ exists, then also $\Gamma\text{-}\lim_{k \rightarrow \infty} F_{n_k}(x)$ exists.

Proof. Let $y_k \rightarrow y_\infty$ be any sequence. Let F_∞ denote the Γ -liminf of $F_n(x)$. Claim

$$\liminf_{k \rightarrow \infty} F_{n_k}(y_k) \geq F_\infty(y_\infty)$$

Let us consider the sequence x_n defined by setting

$$x_i = \begin{cases} y_k & \text{if } i = n_k \text{ for some } k \\ y_\infty & \text{otherwise} \end{cases}$$

n_1	n_2	n_3	n_4	n_5
$\odot$	$\odot$	$\circ$	$\odot$	$\circ$
1	2	3	4	5
y_1	y_2	y_∞	y_3	y_∞
y_1	y_2	y_∞	y_3	y_∞

Clearly $x_n \rightarrow y_\infty$, and therefore

$$\begin{aligned} F_\infty(y_\infty) &\leq \liminf_{n \rightarrow \infty} F_n(x_n) \leq \liminf_{k \rightarrow \infty} F_{n_k}(x_{n_k}) \\ &= \liminf_{k \rightarrow \infty} F_{n_k}(y_{n_k}) \end{aligned}$$

This proves the first inequality. The last one is analogous.

Lemma 2 (For T -liminf)

Assume that $F_\infty(x) = T\text{-}\liminf_{n \rightarrow \infty} F_n(x)$.

Then

$$\forall y \in X, \forall i \in \mathbb{N}, \forall \varepsilon > 0, \forall \tau > 0 \quad \exists m \in \mathbb{N} \quad \exists z \in X \text{ s.t.}$$

$$m \geq i, \quad d(y, z) \leq \tau, \quad F_m(z) \leq F_\infty(y) + \varepsilon$$

Proof. By definition of T -liminf there exists $y_m \rightarrow y$ such that

$$\liminf_{n \rightarrow \infty} F_n(y_m) \leq F_\infty(y) + \frac{1}{2} \varepsilon$$

In particular

$$F_n(y_m) \leq F_\infty(y) + \varepsilon \quad \text{for infinitely many } n\text{'s}$$

Now it is enough to take $m := \text{index large enough for which the relation is true and } z := y_i$
 $\text{---} \circ \text{---} \circ \text{---}$ ↑ m , of course (added after video)

Remark Lemma 2 can be extended more easily to the Γ -limsup.

$\text{---} \circ \text{---} \circ \text{---}$

Lemma 3 (Convergence of minima)

Assume F_∞ is the Γ -liminf.

Then for every compact set $K \subseteq X$ it turns out that

$$\liminf_{n \rightarrow \infty} \inf_{x \in K} F_n(x) \geq \inf_{x \in K} F_\infty(x)$$

Proof. By definition of $\inf$, for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$F_n(x_n) \leq \underbrace{\inf_{x \in K} F_n(x)}_{I_n} + \frac{1}{n}$$

Let us take a subsequence n_k such that

$$\liminf_{n \rightarrow \infty} I_n = \lim_{k \rightarrow \infty} I_{n_k} \quad \text{and} \quad x_{n_k} \rightarrow x_\infty \quad (K \text{ is compact})$$

Then

$$\begin{aligned} \liminf_{n \rightarrow \infty} I_n &= \lim_{k \rightarrow \infty} I_{n_k} \geq \liminf_{k \rightarrow \infty} I_{n_k} \\ &\geq \liminf_{k \rightarrow \infty} \left(F_{n_k}(x_{n_k}) - \frac{1}{n_k} \right) \\ &= \liminf_{k \rightarrow \infty} F_{n_k}(x_{n_k}) \end{aligned}$$

Lemma 1 $\rightarrow \geq F_\infty(x_\infty) \geq \inf_{x \in K} F_\infty(x).$

$\text{---} \circ \text{---} \circ \text{---}$

Theorem 1 Let $F_\infty(x)$ be the Γ -limsup.
Then F_∞ is LSC.

Proof. Let $x_n \rightarrow x_\infty$ be any sequence. We claim that

$$\liminf_{n \rightarrow \infty} F_\infty(x_n) \geq F_\infty(x_\infty)$$

We define a sequence n_k and a sequence y_k as follows:

- select n_1 and y_1 at random
- assume that $n_1, \dots, n_{k-1}$ and $y_1, \dots, y_{k-1}$ have already been selected. Apply Lemma 2 with

$$y := x_k, \quad i := n_{k-1} + 1, \quad \varepsilon := \frac{1}{k}, \quad \eta := \frac{1}{k}$$

We obtain z , which we call y_k , and m , which we call n_k , such that

$$\underbrace{n_k \geq n_{k-1} + 1}_{\{n_k\} \text{ is increasing}}, \quad \underbrace{d(x_k, y_k) \leq \frac{1}{k}}_{y_k \rightarrow x_\infty}, \quad F_\infty(y_k) \leq F_{n_k}(x_k) + \frac{1}{k}$$

$\xrightarrow{\text{swap indices}} F_{n_k}(y_k) \leq F_\infty(x_k) + \frac{1}{k}$
 (added after video)

In conclusion $y_k \rightarrow y_\infty$ and

$$\liminf_{k \rightarrow \infty} F_\infty(y_k) \leq \liminf_{k \rightarrow \infty} F_{n_k}(x_k)$$

$\xrightarrow{\text{Lemma 1}} F_\infty(x_\infty)$
 $\xrightarrow{\text{swap indices}} F_\infty(x_\infty)$
(added after video)

which proves that F_∞ is LSC.

The proof that Γ -limsup is LSC is analogous starting from the second version of Lemma 2.

Convergence of minima and minimizers

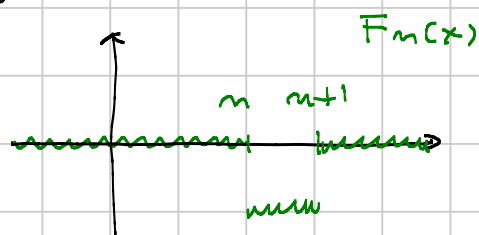
Assume that $F_n \xrightarrow{\Gamma} F_\infty$. In general it is not true that

$$\inf_{x \in X} F_n(x) \rightarrow \inf_{x \in X} F_\infty(x)$$

Example $X = \mathbb{R}$

$$\inf F_n = -1, \quad \inf F_\infty = 0.$$

— 0 — 0 —



Definition The sequence $F_n: X \rightarrow \overline{\mathbb{R}}$ is **EQUICOERCIVE** if there exists a compact set $K \subseteq X$ such that

$$\inf_{x \in X} F_n(x) = \inf_{x \in K} F_n(x) \quad \forall n \in \mathbb{N}.$$

— 0 — 0 —

Theorem Assume that $F_n \xrightarrow{\Gamma} F_\infty$ and F_n is equicoercive. Then

(1) F_∞ has a minimum in X

(2)

$$\inf_{x \in X} F_n(x) \rightarrow \min_{x \in X} F_\infty(x) \quad (\text{convergence of minima})$$

(3) Assume that $x_n \rightarrow x_\infty$ and x_n is a "quasi minimum" for $F_n(x)$, namely

$$\lim_{n \rightarrow \infty} (F_n(x_n) - \inf_{x \in X} F_n(x)) = 0$$

Then $x_\infty \in \arg\min_{x \in X} F_\infty(x)$ (convergence of minimizers)

Proof.

(1) We know that F_∞ is LSC, and therefore it admits a minimum in K . Let $x_0 \in \underset{x \in K}{\operatorname{argmin}} F_\infty(x)$.

We claim that x_0 is min. point in X . Let us take any other $y \in X$. Let $y_n \rightarrow y$ be a recovery sequence. Then

$$\begin{aligned} F_\infty(y) &\geq \limsup_{n \rightarrow \infty} F_n(y_n) \geq \liminf_{n \rightarrow \infty} F_n(y_n) \\ &\quad \uparrow \\ &\quad y_n \text{ recovery} \end{aligned}$$

$$\text{(equicoerciveness)} = \liminf_{n \rightarrow \infty} \sup_{x \in K} F_n(x)$$

$$\text{(Lemma 3)} \geq \sup_{x \in K} F_\infty(x)$$

$$= F_\infty(x_0)$$

(2) We have to prove that $\sup_{x \in X} F_n(x) \rightarrow \min_{x \in X} F_\infty(x)$

$$\text{We observe that } \liminf_{n \rightarrow \infty} \sup_{x \in X} F_n(x) \geq \min_{x \in X} F_\infty(x)$$

follows from Lemma 3 and equicoerciveness.

Let $x_n \rightarrow x_0$ be a recovery sequence for $x_0 \in \underset{x \in X}{\operatorname{argmin}} F_\infty(x)$. Then

$$\limsup_{n \rightarrow \infty} \sup_{x \in X} F_n(x) \leq \limsup_{n \rightarrow \infty} F_n(x_n) \leq F_\infty(x_0) \quad \uparrow \\ \quad \quad \quad x_n \text{ is recovery.}$$

(3) Now $x_n \rightarrow x_\infty$ and $(F_n(x_n) - \inf_{x \in X} F_n(x)) \rightarrow 0$

Then

$$\limsup_{n \rightarrow \infty} F_n(x_n) = \limsup_{n \rightarrow \infty} \left[\sup_{x \in X} F_n(x) + F_n(x_n) - \underbrace{\sup_{x \in X} F_n(x)}_{\downarrow 0} \right]$$

$$= \limsup_{n \rightarrow \infty} \sup_{x \in X} F_n(x)$$

$$(2) = \min_{x \in X} F_\infty(x)$$

On the other hand $\limsup_{n \rightarrow \infty} F_n(x_n) \geq \liminf_{n \rightarrow \infty} F_n(x_n) \geq F_\infty(x_\infty)$

This proves that $F_\infty(x_\infty) \leq \min_{x \in X} F_\infty(x) \Rightarrow x_\infty \text{ is argmin.}$

— 0 — 0 —