

GAMMA CONVERGENCE

Setting: (X, d) metric space
 $F_n: X \rightarrow \bar{\mathbb{R}}$

Def. The sequence F_n Γ -converges to $F_\infty: X \rightarrow \bar{\mathbb{R}}$ if the following two properties are satisfied

(i) (LIMINF INEQUALITY) for every conv. seq. $x_n \rightarrow x_\infty$ in X

$$\liminf_{n \rightarrow \infty} F_n(x_n) \geq F_\infty(x_\infty)$$

(ii) (LIMSUP INEQUALITY) For every $x \in X$, there exists $x_n \rightarrow x$ such that

$$\limsup_{n \rightarrow \infty} F_n(x_n) \leq F_\infty(x_\infty)$$

Defn. Every seq. $x_n \rightarrow x$ satisfying the limsup ineq. is called RECOVERY SEQUENCE for x .

Basic fact A recovery seq. actually satisfies (if $F_n \xrightarrow{\Gamma} F_\infty$)

$$\lim_{n \rightarrow \infty} F_n(x_n) = F_\infty(x_\infty)$$

Proof.

$$F_\infty(x_\infty) \stackrel{(i)}{\leq} \liminf_{n \rightarrow \infty} F_n(x_n) \stackrel{\text{trivial}}{\leq} \limsup_{n \rightarrow \infty} F_n(x_n) \stackrel{(ii)}{\leq} F_\infty(x_\infty).$$

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Property 1 Assume that $F_m(x) = F(x)$ for every $m \in \mathbb{N}$.
Then

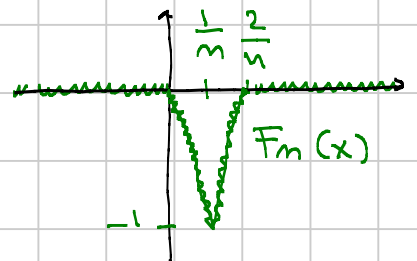
$$F_m \xrightarrow{\Gamma} F$$

(the relaxation is the Γ -limit of the constant sequence $F_m \equiv F$)

Proof. Equivalent to strategy 1 of the verification of relaxation.

Remark This shows that Γ -convergence is not implied by pointwise convergence or uniform convergence.

Example 1 $F_m: \mathbb{R} \rightarrow \mathbb{R}$ as in the picture



Claim

$$F_m \xrightarrow{\Gamma} F_\infty(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Proof. (LIMINF INEQUALITY) $\liminf F_m(x_n) \geq F_\infty(x_\infty)$
for every $x_n \rightarrow x_\infty$

Trivial if $x_\infty = 0$. If $x_\infty \neq 0$, then also $F_m(x_n) = 0$

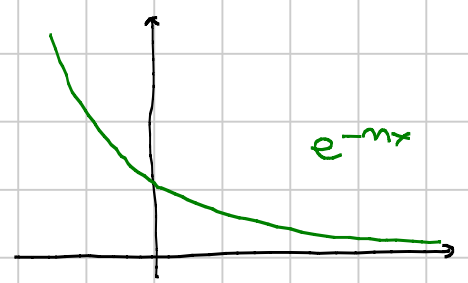
(LIMSUP INEQ) $\forall x \in \mathbb{R} \exists x_n \rightarrow x$ s.t. $\limsup F_m(x_n) \leq F_\infty(x)$

If $x \neq 0$ then take $x_n \equiv x$

If $x = 0$ then take $x_n = \frac{1}{n}$ ← recovery seq. for $x = 0$

Remark In example 1 the pointwise limit exists and it is $\equiv 0$,
and therefore different from the Γ -limit.

Example 2 $F_n: \mathbb{R} \rightarrow \mathbb{R}$ $F_n(x): e^{-nx}$



The pointwise limit is

$$\lim_{n \rightarrow \infty} F_n(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ +\infty & \text{if } x < 0 \end{cases}$$

The Γ -limit is

$$\Gamma\text{-}\lim_{n \rightarrow \infty} F_n(x) = F_\infty(x) := \begin{cases} +\infty & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

(liminf ineq.) $\liminf F_n(x_n) \geq F_\infty(x)$

Trivial for $x \geq 0$, and easy for $x < 0$ (if $x_n \rightarrow x < 0$, then $x_n \leq -\frac{1}{2}x$ eventually)

(limsup ineq.) $\limsup F_n(x_n) \leq F_\infty(x)$

Trivial for $x \neq 0$, because we can take $x_n \equiv x$
For $x = 0$, we can take

$$x_n = \frac{1}{\sqrt{n}}$$

so that $F_n(x_n) = e^{-\sqrt{n}} \rightarrow 0$

Example 3 Assume that

- (i) $F_n(x) \rightarrow F_\infty(x)$ uniformly on compact subsets of X
- (ii) $F_\infty(x)$ is continuous in X

Then $F_n(x) \xrightarrow{\Gamma} F_\infty(x)$

Proof. (limsup inequality) Take $x_n \equiv x$ (pointwise convergence is enough)

(liminf inequality) For every $x_n \rightarrow x_\infty$ assumptions (i) and (ii) guarantee that

$$F_n(x_n) \longrightarrow F_\infty(x_\infty)$$

$$\left[\underbrace{d(F_\infty(x_\infty), F_n(x_n))}_{\longrightarrow 0} \leq \underbrace{d(F_\infty(x_\infty), F_\infty(x_n))}_{\longrightarrow 0} + \underbrace{d(F_\infty(x_n), F_n(x_n))}_{\longrightarrow 0} \right]$$

Property Assume that $F_n \xrightarrow{\Gamma} F_\infty$

Assume that $G: X \rightarrow \bar{\mathbb{R}}$ is continuous

Then $F_n + G \xrightarrow{\Gamma} F_\infty + G$ (stability under const. perturbations)

Proof. (liminf ineq.) Take any $x_n \rightarrow x_\infty$. Then

$$\begin{aligned} \liminf_{n \rightarrow \infty} F_n(x_n) + G(x_n) &\geq \underbrace{\liminf_{n \rightarrow \infty} F_n(x_n)}_{\Gamma\text{-conv.}} + \underbrace{\liminf_{n \rightarrow \infty} G(x_n)}_{G \text{ is LSC}} \\ &\geq F_\infty(x_\infty) + G(x_\infty) \end{aligned}$$

(limsup ineq.) Let $x \in X$ and let $x_n \rightarrow x$ be a rec. seq. for F_n .
Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} F_n(x_n) + G(x_n) &\leq \limsup_{n \rightarrow \infty} F_n(x_n) + \underbrace{\limsup_{n \rightarrow \infty} G(x_n)}_{G \text{ is USC}} \\ &\leq \underbrace{F_\infty(x_\infty)}_{\longrightarrow 0} + \underbrace{G(x_\infty)}_{\longrightarrow 0} \end{aligned}$$

Remark It can be generalized to $F_n \xrightarrow{\Gamma} F_\infty$ and $G_n \rightarrow G_\infty$ unif with G_∞ continuous.

Proof: exercise.

Def. Given $F_n: X \rightarrow \bar{\mathbb{R}}$. One defines

$$\Gamma\text{-}\liminf F_n(x) := \inf \left\{ \liminf_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

$$\Gamma\text{-}\limsup F_n(x) := \inf \left\{ \limsup_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

Proposition • $\Gamma\text{-}\liminf F_n(x) \leq \Gamma\text{-}\limsup F_n(x) \quad \forall x \in \mathbb{R}$

- We have equality for every $x \in X$ iff the Γ -limit exists.

Proof. Trivial from the definitions
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The other side of the moon. What we defined are actually

$\Gamma^+\text{-}\liminf$ and $\Gamma^+\text{-}\limsup$.

Analogous definitions can be made with sup instead of inf.
This leads to

$$\Gamma^+\text{-}\liminf F_n(x) := \sup \left\{ \liminf_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

$$\Gamma^+\text{-}\limsup F_n(x) := \sup \left\{ \limsup_{n \rightarrow \infty} F_n(x_n) : x_n \rightarrow x \right\}$$

Comment Γ^- is suited for min pblms
 Γ^+ is suited for max pblms.
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