

EXTENSION BY RELAXATION

Given: set X and a function $f: X \rightarrow \bar{\mathbb{R}}$
metric space $\hat{X}$ containing X as a subset.

Goal: extending f to a function $\hat{f}: \hat{X} \rightarrow \bar{\mathbb{R}}$

General procedure: consider $f_1: \hat{X} \rightarrow \bar{\mathbb{R}}$

$$f_1(x) = \begin{cases} f(x) & x \in X \\ +\infty & \text{otherwise} \end{cases}$$

Take $\hat{f}$ to be the relaxation of f_1 .

We have for free that

1 - $\hat{f}$ is LSC,

2 - $\hat{f}(x) \leq f(x)$ in X .

Main example $X := C^1([a,b])$

$$F(u) = \int_a^b (\dot{u}^2 + u^2) dx$$

Let $\hat{X} := L^2((a,b))$ or $\hat{X} := C^0([a,b])$

Problem: extending $F(u)$ to $\hat{X}$.

In both cases, the extension is

$$\hat{F}(u) = \begin{cases} \int_a^b (\dot{u}^2 + u^2) dx & \text{if } u \in H^1((a,b)) \\ +\infty & \text{if } u \in \hat{X} \setminus H^1((a,b)) \end{cases}$$

Remark This provides a 3rd defn. of the Sobolev space H^1 , as the set of all u in C^0 or L^2 such that the extension by relaxation is $< +\infty$.

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Strategies for proving that G is the relaxation of F

Strategy 1 ① For every $x_n \rightarrow x_\infty$ it turns out that

(LIMINF INEQ) $\liminf_{n \rightarrow \infty} F(x_n) \geq G(x_\infty) \quad (\bar{F} \geq G)$

② For every $x \in X$ there exists a sequence $x_n \rightarrow x$ such that

$\liminf_{n \rightarrow \infty} F(x_n) \leq G(x) \quad (\bar{F} \leq G)$

In practise, proving ② is equivalent to finding $x_n \rightarrow x$ s.t.

(LIMSUP INEQ) $\limsup_{n \rightarrow \infty} F(x_n) \leq G(x)$

More important: it is enough to consider $x \in D$, where D is dense in energy wrt G .

Strategy 2

- ① G is L.S.C.
- ② $G(x) \leq F(x) \quad \forall x \in X$
- ③ The existence of a recovery sequence for every $x \in D$ (dense in energy)

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Proof of the example Want to find relaxation in $L^2(a,b)$

$$F(u) = \begin{cases} \int_a^b u^2 + u^2 & \text{if } u \in C^1 \\ +\infty & \text{otherwise} \end{cases}$$

Claim

$$\bar{F}(u) = G(u) = \begin{cases} \int_a^b u^2 + u^2 & \text{if } u \in H^1 \\ +\infty & \text{otherwise} \end{cases}$$

LIMINF INEQUALITY Given any $u_m \xrightarrow{L^2} u_\infty$, prove that

$$\liminf F(u_m) \geq G(u_\infty)$$

We can assume WLOG that $LHS < +\infty$. This implies that there exists u_{m_k} and M such that

$$F(u_{m_k}) \leq M \quad \forall k \in \mathbb{N}$$

$$\Rightarrow \|u_{m_k}\|_2^2 \leq M \Rightarrow \text{up to further subsequences}$$

$$u_{m_k} \rightarrow \dot{v} = \dot{u}_\infty$$

This implies that $u_\infty \in H^1(a,b)$. At this point we can pass to the liminf as usual

$$\liminf_{k \rightarrow \infty} F(u_{m_k}) = \liminf \int_a^b (\dot{u}_{m_k}^2 + u_{m_k}^2) \geq \int_a^b \dot{u}_\infty^2 + u_\infty^2$$

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LSC of the
work.

This proves the liminf ineq. up to choosing m_k in such a way that

$$\liminf_{k \rightarrow \infty} F(u_{m_k}) = \liminf_{n \rightarrow \infty} F(u_n)$$

LIMSUP INEQ. For every $u \in L^2$ we need a sequence $u_n \rightarrow u$ such that

$$\limsup_{n \rightarrow \infty} F(u_n) \leq G(u)$$

This is trivial if $u \notin H^1$ because $G = +\infty$ there. So assume that $u \in H^1$. By definition (H) of Sobolev spaces

$$\exists \{u_n\} \subseteq C^\infty \text{ s.t. } u_n \rightarrow u \text{ and } u_n \rightarrow u.$$

This is exactly what we need.

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Remark. Proving by strategy 1 that G is the relaxation of F implies as a by-product that G is LSC.

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Very important example (not discussed here). Let us consider a geometric problem, for example in $\mathbb{R}^2$

$$F(A) = \int_A f(x) dx + \text{Per}(A)$$



where $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ (for example continuous), $A \subseteq \mathbb{R}^2$ is a bounded subset, and $\text{Per}(A)$ is the perimeter of A .



Question: What is the perimeter?

A way to define the perimeter for non-regular sets A is the following

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- Define X as the set of polygons (where Per is a classical notion)
- Define $\hat{X}$ as the set of bounded measurable subsets of $\mathbb{R}^2$, which is a metric space w.r.t to L^1 distance of charact. functions
- Extend by relaxation Per from X to $\hat{X}$.

Example

$$\inf \left\{ \underbrace{\int_a^b [\sin(u) + (u-x)^2] dx}_{F(u)} : u \in C^{2,2}([a,b]) \right\}$$

Claim: $\bar{F}(u) = \int_a^b (u-x)^2 dx - (b-a)$

If the claim is true, then $\inf = -(b-a)$ because

$$\inf F = \inf \bar{F}$$

In turn, the claim is equivalent to showing that

$$G(u) := \int_a^b \sin(u) \quad \text{has relaxation} \quad \bar{G}(u) := \int_a^b -1 = -(b-a)$$

because the second term $\int_a^b (u-x)^2 dx$ is continuous.

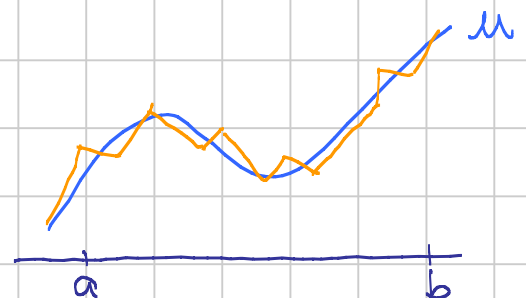
In order to prove that $\bar{G}$ is the relaxation of G , the limiting ineq. is trivial because

$$G(u_n) \geq \int_a^b (-1) \quad \text{for every } u_n.$$

It remains to prove that there exists a recovery sequence, namely every u can be approximated, for example in C^0 or L^2 , by a sequence u_n with $\sin(u_n) = -1$.

It is enough that u_n is piecewise affine and

$$u_n(x) \in \left\{ \frac{3\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$$



for every $x \in [a,b]$ (on each subinterval)

Remark Enough to do this for every $u \in C^1$ or C^∞ or any other $\mathcal{D}$ dense in energy.