

Calculus of Variations — LECTURE 20

Note Title

28/10/2016

RELAXATION AND MIN. PBMS

General setting: (X, d) metric space, $f: X \rightarrow \bar{\mathbb{R}}$, $\bar{f}: X \rightarrow \bar{\mathbb{R}}$
relaxation of f .

Property 1 $\sup \{f(x) : x \in X\} = \sup \{\bar{f}(x) : x \in X\}$

Proof. Assume $\bar{I}$ is the 2nd sup and $\bar{I} \in \mathbb{R}$ (the other case is simpler).

By defn. of sup, there exists $x \in X$ s.t.

$$\bar{f}(x) \leq \bar{I} + \varepsilon$$

By the usual lemma, there exists a point $y \in X$ (actually as close as we want to x) such that

$$f(y) \leq \bar{f}(x) + \varepsilon \leq \bar{I} + 2\varepsilon$$

This implies that $\sup_{x \in X} f \leq \sup_{x \in X} \bar{f}$

The reverse inequality is trivial because $\bar{f} \leq f$.

Defn We say that f is coercive if there exists a compact set $K \subset X$ s.t.

$$\sup_{x \in K} f(x) = \sup_{x \in X} f(x)$$

Property 2 Assume that f is coercive.

Then

$$\min \{ \bar{f}(x) : x \in X \} \quad (= \inf \{ f(x) : x \in X \})$$

exists.

Proof. We already know that $\bar{f}$ is LSC, and hence it admits the minimum in K . We claim that the min in K is actually also the min in X .

Let I denote the inf of f in K (or in X).

Then there exists a sequence $\{x_n\} \subseteq K$ such that

$$f(x_n) \rightarrow I$$

Up to subsequences we can assume that $x_n \rightarrow x_\infty$ (because K is compact). Claim: x_∞ is a min. point of $\bar{f}(x)$.

Let $y \in X$ be any point, and let $y_n \rightarrow y$ be a recov. seq. for y . Then

$$\begin{array}{ccccccc} \bar{f}(y) & = & \lim_{n \rightarrow \infty} f(y_n) & \geq & I & = & \lim_{n \rightarrow \infty} f(x_n) \geq \bar{f}(x_\infty) \\ \uparrow \text{ } y_n \text{ is recovery} & & & & \uparrow \text{ } I = \inf_{x \in X} & \downarrow \text{ defn. of } x_n & \uparrow \text{ defn. of } \bar{f} \end{array}$$

Property 3 Assume again that f is coercive. Let $\{x_n\} \subseteq K$ be any sequence such that

$$f(x_n) \rightarrow \inf_{x \in X} f(x) \quad (\text{min. sequence})$$

Then any limit point of x_n is a min point of the relaxation $\bar{f}$.

Proof: see above

Remark If there exists $M > I$ such that

$$\{x \in X : f(x) \leq M\}$$

is compact, then any minimizing sequence is relatively compact (it lies in the sublevel) and all its limit points are min. points of $\bar{f}(x)$.

Property 4 Assume as in the remark and in addition assume that $\bar{f}$ has a UNIQUE min. point x_0 . Then the set of min. seq. for f coincides with the set of recovery sequence for x_0 .

$$\begin{array}{ccc} \text{---} & \circ & \text{---} \\ \text{---} & \circ & \text{---} \end{array}$$

Defn. Let $f: X \rightarrow \bar{\mathbb{R}}$ be any function.

A subset $D \subseteq X$ is called DENSE IN ENERGY (wrt f) if

$$\forall x \in X \exists \{x_n\} \subseteq D \text{ s.t. } x_n \rightarrow x \text{ and } f(x_n) \rightarrow f(x)$$

Prop. D is dense in energy iff

$$\{(d, f(d)) : d \in D\}$$

is dense in the graph (wrt to $X \times \mathbb{R}$)

Example If f is cont., any $D \subseteq X$ dense is dense in energy.

Without continuity this is false.

Lemma Let $\varphi: X \rightarrow \mathbb{R}$ and $\psi: X \rightarrow \mathbb{R}$ be two functions
Let $D \subseteq X$.

Let us assume that

- (i) D is dense in energy for ψ ,
- (ii) φ is LSC,
- (iii) $\varphi(x) \leq \psi(x)$ for every $x \in D$.

Then

$$\varphi(x) \leq \psi(x) \quad \forall x \in D.$$

Proof. Let $x \in X$ be any point. Let $\{x_n\} \subseteq D$ be as in defn. of the subset dense in energy.
Then

$$\varphi(x) \leq \liminf \varphi(x_n) \leq \liminf \psi(x_n) = \psi(x)$$

$\uparrow$ φ is LSC $\uparrow$ (iii) $\uparrow$ D is dense in energy for ψ .

Question: assume that $f: X \rightarrow \overline{\mathbb{R}}$ is given.

Assume that we suspect that $g: X \rightarrow \overline{\mathbb{R}}$ is the relaxation of f .

How to prove it?

First strategy Need to check 3 things

- 1 $f(x) \geq g(x)$ for every $x \in X$
- 2 $g(x)$ is LSC
- 3 there exists $D \subseteq X$ which is dense in energy for g such that

$$\forall x \in D \quad \exists x_n \rightarrow x \text{ s.t. } f(x_n) \rightarrow g(x)$$

(existence of a recovery for every $x \in D$)

Proof $\bar{f}(x) \geq g(x)$ Trivial because $\bar{f} = \hat{f} \geq g$

$\bar{f}(x) \leq g(x)$ Due to [3] we know that

$$\bar{f}(x) \leq \liminf_{m \rightarrow \infty} f(x_m) = g(x) \quad \forall x \in D$$

$\uparrow$ defn of $\bar{f}$ $\uparrow$ [3]

Now we apply the lemma with

$\varphi(x) := \bar{f}(x)$
 $\uparrow$
LSC because
it's a relaxation

$\psi(x) := g(x)$
 $\uparrow$
admits a dense in energy
where $\varphi \leq \psi$

$\Rightarrow \bar{f}(x) \leq g(x)$ for every $x \in X$.
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