

Preliminaries about LSC functions

General setting: (X, d) metric space, $f: X \rightarrow \bar{\mathbb{R}}$ is a function

Prop. (characterization of LSC)

The following facts are equivalent

(i) For every $x_0 \in X$ which is not isolated

$$\liminf_{y \rightarrow x} f(y) \geq f(x_0),$$

(ii) For every conv. sequence $x_n \rightarrow x_\infty$

$$\liminf_{n \rightarrow \infty} f(x_n) \geq f(x_\infty),$$

(iii) For every $M \in \mathbb{R}$ the sublevel

$$\{x \in X : f(x) \leq M\} \text{ is closed}$$

(iv) The EPIGRAPH

$$\{(x, y) \in X \times \mathbb{R} : y \geq f(x)\}$$

is closed in $X \times \mathbb{R}$.

Proof. Exercise.

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Def. (LSC envelope). Let $f: X \rightarrow \bar{\mathbb{R}}$ be any function.

The LSC envelope is the function

$$\hat{F}(x) = \sup \left\{ g(x) : g: X \rightarrow \bar{\mathbb{R}} \text{ LSC and } g(y) \leq f(y) \forall y \in X \right\}$$

(supremum of all LSC functions below f)

Prop. $\hat{F}$ is LSC. More generally: the supremum of any family of LSC functions is LSC.

Proof. Let $f_i: X \rightarrow \bar{\mathbb{R}}$ for every $i \in I$. Let

$$f(x) := \sup \{ f_i(x) : i \in I \}$$

Assume f_i is LSC for every $i \in I$.

Two ways

- with sublevels $\{x \in X : f(x) \leq M\} = \bigcap_{i \in I} \{x \in X : f_i(x) \leq M\}$

- the same with epigraphs.

Def. Given $f: X \rightarrow \bar{\mathbb{R}}$, the relaxation is the function

$$\bar{f}(x) := \inf_{n \rightarrow \infty} \left\{ \liminf_{m \rightarrow \infty} f(x_m) : x_m \rightarrow x \right\}$$

Remark Why we do not define

$$\bar{f}(x) := \liminf_{y \rightarrow x} f(y)$$

FALSE

Big difference: constant sequence $x_m \equiv x$ is allowed in the defn. of $\bar{f}(x)$ but not in computing the $\liminf$

Lemma Assume $\bar{f}(x)$ is the relaxation of $f(x)$.

Then

$\forall x \in X, \forall \varepsilon > 0, \forall r > 0 \exists y \in X$ such that if $\bar{f}(x) \in \mathbb{R}$

$$d(x, y) \leq r \quad \text{and} \quad f(y) \leq \bar{f}(x) + \varepsilon$$

Proof. By definition of infimum, there exists a sequence $x_n \rightarrow x$ such that

$$\liminf_{n \rightarrow \infty} f(x_n) \leq \bar{f}(x) + \frac{1}{2} \varepsilon$$

In particular this implies that

$$f(x_n) \leq \bar{f}(x) + \varepsilon \quad (*)$$

for infinitely many indices n . Now we can take $y := x_n$ for n large enough and among the indices for which $(*)$ holds.

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Property 1 $\bar{f}(x) \leq f(x) \quad \forall x \in X$

Proof Trivial: consider $x_n \equiv x$

Property 2 The infimum in the defn of $\bar{f}(x)$ is actually a min, namely there exists a sequence $x_n \rightarrow x$ such that

$$\bar{f}(x) = \lim_{n \rightarrow \infty} f(x_n)$$

Proof: Let us assume that $\bar{f}(x) \in \mathbb{R}$ (other cases are simpler).
Apply the lemma with

$$\varepsilon := \frac{1}{n}$$

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$\leadsto$ obtain a point $x_n \in X$ s.t.

$$\underbrace{d(x, x_n) \leq \frac{1}{n}}_{x_n \rightarrow x}$$

$$\underbrace{f(x_n) \leq \bar{f}(x) + \frac{1}{n}}_{\downarrow}$$

$$\limsup_{n \rightarrow \infty} f(x_n) \leq \bar{f}(x)$$

On the other hand

$$\bar{f}(x) \leq \liminf_{n \rightarrow \infty} f(x_n) \leq \limsup_{n \rightarrow \infty} f(x_n) \leq \bar{f}(x)$$

In other words, $f(x_n) \rightarrow \bar{f}(x)$
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Definition Let $\bar{f}(x)$ be the relaxation of $f(x)$.

Every sequence $x_n \rightarrow x$ s.t. $f(x_n) \rightarrow \bar{f}(x)$ is called a RECOVERY SEQUENCE for x .

Property 2 restated: every $x \in X$ admits a recovery sequence
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Property 3 $\bar{f}$ is LSC.

Proof. Let us take any sequence $x_n \rightarrow x_\infty$. We need to prove that

$$\liminf_{n \rightarrow \infty} \bar{f}(x_n) \geq \bar{f}(x_\infty)$$

For every $n \in \mathbb{N}$ let us apply the lemma with base point x_n

$$\varepsilon := \frac{1}{n}$$

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$\leadsto$ obtain a point $y_n \in X$ such that

$$d(y_n, x_n) \leq \frac{1}{n}$$

$\Downarrow$

$$y_n \rightarrow x_\infty$$

$$f(y_n) \leq \bar{f}(x_n) + \frac{1}{n}$$

$\Downarrow$

$$\liminf_{n \rightarrow \infty} f(y_n) \leq \liminf_{n \rightarrow \infty} \bar{f}(x_n)$$

We conclude by remarking that $\bar{f}(x_\infty) \leq \liminf_{n \rightarrow \infty} f(y_n)$.

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Property 4 The relaxation of f coincides with the LSC envelope of f .

Proof. $\bar{f} := \text{relaxation}$ $\hat{f} : \text{LSC envelope}$

$$\bar{f} \leq \hat{f}$$

Trivial, because we know that $\bar{f}$ is LSC and below f , so it competes in the defn. of $\hat{f}$.

$$\bar{f} \geq \hat{f}$$

We need that every $g: X \rightarrow \bar{\mathbb{R}}$ which is LSC and below f is actually below $\bar{f}$. Let us take any such g , and any $x \in X$.

Let $x_n \rightarrow x$ be any recovery sequence for x

$$g(x) \leq \liminf_{n \rightarrow \infty} g(x_n) \leq \liminf_{n \rightarrow \infty} f(x_n) = \bar{f}(x)$$

$\uparrow$ $g \text{ is LSC}$ $\uparrow$ $g \leq f$ $\uparrow$ $x_n \text{ is a recovery.}$

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Property 5 (Stability under continuous perturbations)

Assume $f: X \rightarrow \bar{\mathbb{R}}$ and let $g: X \rightarrow \bar{\mathbb{R}}$ continuous. Then

$$\overline{f+g} = \bar{f} + g$$

Proof. Let $x_n \rightarrow x$ be any sequence.

$$\begin{aligned}\liminf_{n \rightarrow \infty} (f(x_n) + g(x_n)) &\geq \liminf_{n \rightarrow \infty} f(x_n) + \liminf_{n \rightarrow \infty} g(x_n) \\ &\geq \bar{f}(x) + g(x)\end{aligned}$$

This proves that $\overline{f+g}(x) \geq \bar{f}(x) + g(x)$ (requires just that g is LSC)

Now let $x_n \rightarrow x$ be a recovery seq. for x for f . Then

$$\begin{aligned}\liminf_{n \rightarrow \infty} (f(x_n) + g(x_n)) &\leq \limsup_{n \rightarrow \infty} (f(x_n) + g(x_n)) \\ &\leq \limsup_{n \rightarrow \infty} f(x_n) + \limsup_{n \rightarrow \infty} g(x_n) \\ &\quad \quad \quad \nearrow \text{limit} \\ &\leq \bar{f}(x) + g(x) \quad (\text{enough that } g \text{ is usc})\end{aligned}$$

This proves that $\bar{f}(x) + g(x) \geq \overline{f+g}(x)$, and actually that

$x_n \rightarrow x$ is a recovery also for the sum.
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