

Comments about uniqueness

$$\ddot{u} = f(u) + BC$$

If f is strictly increasing $\leadsto$ existence and uniqueness through var. approach

$$\begin{cases} \ddot{u} = -\lambda \sin u \\ u(0) = 0 \\ u(1) = 0 \end{cases} \quad \text{Easy: } u(x) \equiv 0 \text{ is a solution for every } \lambda \in \mathbb{R}$$

Claim: if λ is large enough, then the solution is NOT unique.

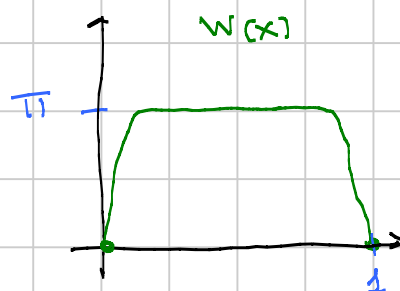
Strategy: consider $\min \left\{ \underbrace{\int_0^1 \frac{1}{2} \dot{u}^2 + \lambda \cos u}_{F(u)} : u(0) = 0 = u(1) \right\}$

$F(0) = \lambda$. I claim that λ is not the minimum of $F(u)$ if λ is large enough

$$F(w) = \int_0^1 \frac{1}{2} \dot{w}^2 + \lambda \cos(w) \stackrel{?}{<} \lambda$$

$$\int_0^1 \frac{1}{2} \dot{w}^2 \stackrel{?}{<} \lambda \int_0^1 (1 - \cos w) dx$$

For a given w , this is a strictly positive number



Thus for λ large $F(w) < F(0) \Rightarrow u(x) \equiv 0$ is not the minimum.

Direct method $\Rightarrow$ min point exists and solves the initial BVP.

Remark $u(x) \equiv 0$ solves (EE) but it is not the global minimum.

Example $\ddot{u} = u^3 + \cos t$

Claim: existence of at least one periodic solution

Remark: RHS is superlinear (cubic growth) $\Rightarrow$ there is no guarantee that a global solution exists.

Strategy Consider

$$F(u) := \int_0^{2\pi} \left[\frac{1}{2} \dot{u}^2 + \frac{1}{4} u^4 + \cos x \cdot u \right] dx$$

and consider

$$\min \{ F(u) : u(0) = u(2\pi) \}$$

Claim: the min pbm has a (unique) solution of class C^∞ which satisfies the equ. with

$$u(0) = u(2\pi) \quad \leftarrow \text{given}$$

$$\dot{u}(0) = \dot{u}(2\pi) \quad \leftarrow \text{generated}$$

Thus if we extend u by periodicity we have solved the initial pbm.

Easy part: every min point satisfies EE + 2BC
(usual $F(u+tv) = \dots$)

Almost easy part : existence through direct method.

Weak formulation : standard

LSC : standard if $i_n \rightarrow i_\infty$ and $u_n \rightarrow u_\infty$ unif.

Regularity + uniqueness : standard.

Remark Uniqueness to the min. pbm. $\nRightarrow$ unique of per. solution

Proof. $\text{cost} = \ddot{u} - u^3$ if u is T -periodic, then cost is T -periodic

$$\Rightarrow T = 2\pi \cdot k \text{ for some } k \geq 1, k \in \mathbb{N}$$

As far as we know, a solution with min. period $= 4\pi$ might exist.

Compactness Obtaining bounds on u and $\ddot{u}$ from bounds on $F(u)$

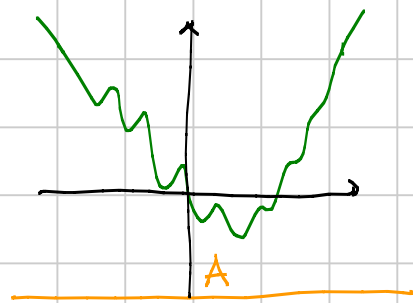
$$M \geq F(u) = \int_0^{2\pi} \frac{1}{2} \ddot{u}^2 + g(x, u) dx \geq \int_0^{2\pi} \frac{1}{2} \ddot{u}^2 + C$$

If $g(x, u)$ is bounded from below, then we obtain a bound on $\|\ddot{u}\|_2$.

In our case

$$g(x, u) = \frac{1}{4} u^4 + \cos x \cdot u$$

$$\geq \frac{1}{4} u^4 - |u| \geq A$$



$\leadsto$ bound on $\|\ddot{u}\|_2$

In order to obtain a bound on u , we observe that

$$\frac{1}{4} u^4 + \cos x \cdot u \geq \frac{1}{40} u^4 - A \quad \forall u \in \mathbb{R}$$

for a suitable A (trivial calculus as before). So

$$M \geq F(u) \geq \int_0^{2\pi} \frac{1}{40} u^4 - A$$

$$\Rightarrow \int_0^{2\pi} u^4 \leq 40M + 40A \cdot 2\pi$$

Integral bound on $u^4 \Rightarrow$ bound on $u^4(x_n)$
 $\Rightarrow$ unif. bound on $u(x)$.
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General discussion about compactness

$$L(x, s, p) := \varphi(p) + \psi(x, s)$$

$$\boxed{\text{Case 1}} \quad \left. \begin{array}{l} \psi(x, s) \geq -A \\ \varphi(p) \geq \underset{\substack{\uparrow \\ \text{positive}}}{B} p^2 - C \end{array} \right\} \Rightarrow \text{bound on } \|\ddot{u}\|_2$$

$$M \geq F(u) = \int_a^b \varphi(u) + \psi(x, u) \geq B \int_a^b \ddot{u}^2 - (A+C)(b-a)$$

$$\Rightarrow \|\ddot{u}\|_2^2 \leq \frac{M + (A+C)(b-a)}{B}$$

$$\boxed{\text{Case 2}} \quad \left. \begin{array}{l} \varphi(p) = p^2 \\ \psi(x, s) \geq A - B|s| \\ \text{DBC} \end{array} \right\} \stackrel{?}{\Rightarrow} \text{bound on } \|\ddot{u}\|_2$$

$$\boxed{\text{YES}} \quad |u(x)| \leq |u(a)| + |u(x) - u(a)| \\ \leq |u(a)| + \underbrace{\|\ddot{u}\|_2 |b-a|^{1/2}}_{\frac{1}{2} - \text{Hölder continuity}}$$

$$\Rightarrow \int_a^b |u(x)| \leq |u(a)|(b-a) + \|\dot{u}\|_2 (b-a)^{1/2} \cdot (b-a)$$

$$= C_1 + C_2 \|\dot{u}\|_2$$

Therefore

$$\begin{aligned} F(u) &= \int_a^b \dot{u}^2 + \psi(x, u) \geq \int_a^b \dot{u}^2 + A - B|u| \\ &= \|\dot{u}\|_2^2 + A(b-a) - B \int_a^b |u| \\ &\geq \|\dot{u}\|_2^2 + A(b-a) - B(C_1 + C_2 \|\dot{u}\|_2) \\ &\geq \underbrace{\|\dot{u}\|_2^2}_{\text{quadratic}} - C_3 - C_4 \underbrace{\|\dot{u}\|_2}_{\text{linear}} \end{aligned}$$

$$F(u) \leq M \Rightarrow \underbrace{\|\dot{u}\|_2^2 - C_4 \|\dot{u}\|_2 - C_3}_{\substack{\text{second order pol.} \\ \text{with pos. coeff. of leading term.}}} \leq M \Rightarrow \|\dot{u}\|_2 \leq \text{const.}$$

Case 3 Analogous argument work if

$$\varphi(p) \geq Ap^2 - B$$

$$\psi(x, s) \geq -C - D|s|^a \quad \text{for some } a < 2$$

Remark 1 $F(u) = \int \dot{u}^2 - u^3$ no bound is possible

$$F(nw) = n^2 \int \dot{w}^2 - w^3 \int w^3$$

$$\leadsto \sup = -\infty$$



Remark 2 If we are as in case 3, but with $a=2$

Then constants come into play, namely the possibility
to obtain a bound depends on $A, D, (b-a)$
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