

Variational approach to boundary value problems

Reverse engineering: given a differential equation with BC, use variational tools in order to prove existence and/or uniqueness.

Example

$$\begin{cases} u'' = \arctan u \\ u(0) = 1 \\ u(2016) = 0 \end{cases} \quad] \text{ DBC}$$

Goal: discussing existence, uniqueness and regularity

Technique: finding a min pbm. whose Euler equation is the given one.

$$(L_\varphi)' = L_s$$

$$L_\varphi = \dot{u}^2$$

$$L_s = \arctan u$$

Let us take $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $g'(u) = \arctan u$ and for example $g(0) = 0$, and consider

$$\min \left\{ \int_0^{2016} \left[\frac{1}{2} \dot{u}^2 + g(u) \right] dx : u(0) = 1, u(2016) = 0 \right\}$$

$$L(x, s, p) := \frac{1}{2} p^2 + g(s)$$

We apply the direct method in order to obtain existence.



1 Weak formulation in $H^1((0, 2016))$

$$F(u) = \int_0^{2016} \frac{1}{2} \dot{u}^2 + \underbrace{g(u)}_{\geq 0} \quad \text{if } u \in H^1((0, 2016)) \text{ and } u \text{ satisfies DBC}$$

2 Compactness (of sublevels of F)

$$F(u) \leq M \Rightarrow \frac{1}{2} \int_0^{2016} |\dot{u}|^2 \leq M \Rightarrow \|\dot{u}\|_2^2 \leq 2M$$

If $\{u_n\}$ is any sequence with $F(u_n) \leq M$, then

$$\|\dot{u}_n\|^2 \leq \sqrt{2M} \quad \text{DBC} \Rightarrow |u_n(x)| \leq M, \quad \forall x \in [0, 2016] \\ \forall n \in \mathbb{N}$$

$\uparrow$
one D condition
is enough

In the usual way, up to subsequences

$$u_n \rightarrow u_\infty \text{ unif.} \quad \dot{u}_n \rightharpoonup \dot{u}_\infty \text{ weakly in } L^2$$

3 Lower semicontinuity Assume that $u_n \rightarrow u_\infty$ as above.
Then

$$\int_0^{2016} g(u_n) \rightarrow \int_0^{2016} g(u_\infty) \quad (\text{due to unif. convergence})$$
$$\liminf_{n \rightarrow \infty} \int_0^{2016} \dot{u}_n^2 \geq \int_0^{2016} \dot{u}_\infty^2$$

$\Rightarrow F$ is LSC with respect to the notion of conv.

1 + 2 + 3 $\Rightarrow$ the minimum problem admits at least one solution in $H^1((0, 2016))$.

Remark Boundary cond. are stable wrt the convergence.

4 Regularity and Euler equation

$$\psi(t) := F(u+tv) \quad v \in C_c^\infty((0, 2016))$$

$$\psi'(t) = \frac{d}{dt} \int_0^{2016} \frac{1}{2} \dot{u}^2 + \frac{1}{2} t^2 \dot{v}^2 + t \dot{u} \dot{v} + g(u+tv)$$

$$= \int_0^{2016} \underbrace{t \dot{v}^2 + \dot{u} \dot{v} + \arctan(u+tv) v}_{\varphi(x,t)}$$

We can exchange integral/derivative provided that $\varphi(x,t)$ is dominated by some integrable function for $t \in (-\delta, \delta)$. This follows from $\dot{u} \in L^2$ and v smooth and $\arctan$ bounded.

$$\psi'(0) = \int_0^{2016} \dot{u} \dot{v} + \arctan u \cdot v = 0 \quad \text{if } u \text{ is any min. point}$$

This can be rewritten as

$$\int_0^{2016} \underbrace{\dot{u} \dot{v}}_f = - \int_0^{2016} \underbrace{\arctan u \cdot v}_g \quad \forall v \in \dots$$

Equivalent to saying that g is the weak deriv. of f . In particular

$$\dot{u} \in H^1 \quad \text{and hence } u \in H^2$$

and

$$\ddot{u} = \arctan u$$

↑
weak second derivative

Usual BOOTSTRAP argument $\rightarrow u$ is of class C^∞ and satisfies BCs.

Uniqueness In this case the solution to the original pblm is unique

1st approach : variational

- Two steps :
- 1 - Any solution to the initial BVP is a minimum point for $F(u)$
 - 2 - $F(u)$ admits a unique min. point.

[2] is easy because $L(s,p)$ is STRICTLY convex in the pair (s,p)

[1] Again convexity. More precisely : every solution to BVP satisfies the first out. form of Euler Egn

$$\int_0^{2016} [\ddot{u} \ddot{v} + \arctan u \cdot v] dx = 0$$

This implies that $u \in C^\infty$ and because of convexity it is a min. point (see lectures about indirect method)

Remark We actually proved that any solution in H^2 to the BVP is actually C^∞ .

2nd approach : ODE argument

Assume that u and v are two solutions, and set $w := u - v$

$$\ddot{u} = \arctan u$$

$$\ddot{v} = \arctan v$$

$$\ddot{w} = \arctan u - \arctan v$$

Let us multiply by $u - v$:

$\arctan x$ is increasing

$$w \ddot{w} = (\arctan u - \arctan v)(u - v) \stackrel{\downarrow}{\geq} 0$$

So $w\ddot{w} \geq 0$ for every $x \in [0, 2016]$, and therefore

$$0 \leq \int_0^{2016} w\ddot{w} dx = \underbrace{[w\dot{w}]_0^{2016}}_{\substack{0 \\ \text{because } w=0 \text{ at the boundary}}} - \int_0^{2016} \dot{w}^2$$

because $w=0$ at the boundary

$$\Rightarrow \int_0^{2016} \dot{w}^2 = 0 \Rightarrow \dot{w} \equiv 0 \Rightarrow w \equiv 0$$

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Example 2
$$\begin{cases} u'' = \arctan u \\ u(0) = 3 & \text{DBC} \\ \dot{u}(2016) = 0 & \text{NBC} \end{cases}$$

As before with $\min \{F(u) : u(0) = 3\}$.

At the end we obtain $\dot{u}(2016)$ as a by-product of Euler equ.

Example 3
$$\begin{cases} \ddot{u} = \arctan u \\ \dot{u}(0) = 0 \\ \dot{u}(2016) = 0 \end{cases}$$

$\leadsto \min F(u)$ without further conditions

$\leadsto$ in this case $u \equiv 0$ is the UNIQUE solution

Example 4
$$\begin{cases} \ddot{u} = \arctan u \\ \dot{u}(0) = 3 \\ \dot{u}(2016) = 3 \end{cases}$$

Two possible approaches

1 Variable change : $u(x) = 3x + v(x)$. What is the equ for v ?

$$\begin{cases} \ddot{v} = \arctan(v+3x) \\ \dot{v}(0) = 0 \\ \dot{v}(2016) = 0 \end{cases}$$

$$\min \left\{ \int_0^{2016} \left[\frac{1}{2} \dot{v}^2 + g(v+3x) \right] dx \right\}$$

Direct method $\leadsto$ solution $\leadsto$ uniqueness and regularity

2 Modify the Lagrangian

$$(L_p)' = L_s \quad L_p = 0 \text{ at the boundary}$$

$$L(x, s, p) = \frac{1}{2} p^2 - 3p + g(s)$$

$$\begin{aligned} \text{Euler equation becomes} \quad (\dot{u} - 3)' &= \arctan u \\ \ddot{u} &= \arctan u \end{aligned}$$

$$\text{BCs generated by min. procedure} \quad L_p = 0, \quad \dot{u} - 3 = 0 \\ \dot{u} = 3 \text{ at boundary.}$$

Remark Compactness is ok because L has quadratic growth in p .

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