

Calculus of Variations - LECTURE 16

Note Title

21/10/2016

Compactness theorems

Almost any compactness theorem in function spaces has 2 main ingredients:

- something that prevents functions from becoming too big
- " " " too many oscillations.

Examples

$u_n(x) := n$ (no oscillations, but no control on functions)

$u_n(x) := \sin(nx)$ (bounded functions, but no control on oscillations).

A bound on derivatives usually gives a bound on oscillations.

Typical result Let $u_n : [a, b] \rightarrow \mathbb{R}$.

Let us assume that

(i) $\|u_n'\|_{L^2([a, b])} \leq M$ ($u_n \in H^1([a, b])$)

(ii) $\forall n \in \mathbb{N} \exists x_n \in [a, b] \text{ s.t. } |u_n(x_n)| \leq M$

Then there exists a subsequence such that

$$u_{n_k} \rightarrow u_\infty \text{ unif.}$$

$$u'_{n_k} \rightarrow u'_\infty$$

Proof. Follows from A.A. where

→ equicontinuity follows from (i)

→ equiboundedness follows from (i) and (ii)

bounded by (ii) bounded due to (i)

$$|u_n(x)| \leq \underbrace{|u_n(x_n)|}_{\text{bounded by (ii)}} + \underbrace{|u_n(x) - u_n(x_n)|}_{\text{bounded due to (i)}} \quad \square$$

What gives (ii) in the case of functionals

→ a DBC in an endpoint

→ some integral constraint such as $\int_a^b u_n dx = \text{constant}$

or $\int_a^b |u_n(x)|^p dx = \text{constant}$

Proof. Use mean value theorem for integrals

$$(b-a) u_n(x_n) = \int_a^b u_n(x) dx = \text{constant}$$

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Lower semicontinuity

Terms involving u

Assume that $u_n \rightarrow u_\infty$ unif. and $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Then

$$\lim_{n \rightarrow +\infty} \int_a^b \psi(u_n(x)) dx = \int_a^b \psi(u_\infty(x)) dx$$

Proof. Enough to show that $\psi(u_n) \rightarrow \psi(u_\infty)$ unif. in $[a,b]$. This exploits the unif. cont. of ψ on compact sets.

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More generally, we can treat in the same ways also

$$\int_a^b \psi(x, u_n(x)) dx \quad \psi: [a,b] \times \mathbb{R} \rightarrow \mathbb{R}.$$

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More general case: assume that $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is just LSC and bounded from below. Then

$$\liminf_{n \rightarrow +\infty} \int_a^b \psi(u_n(x)) dx \geq \int_a^b \psi(u_\infty(x)) dx$$

[Proof: Fatou's Lemma]

Terms involving $\dot{u}$

Before we exploited that $\liminf \int_a^b \dot{u}_n^2 \geq \int_a^b \dot{u}_\infty^2$
provided that $\dot{u}_n \rightarrow \dot{u}_\infty$.

Proof (again): assume that $v_n \rightarrow v_\infty$. Then

$$\begin{aligned} \int_a^b \dot{v}_n^2 &= \int_a^b [v_\infty - (v_\infty - v_n)]^2 \\ &= \int_a^b v_\infty^2 - \underbrace{2 \int_a^b v_\infty (v_\infty - v_n)}_{\downarrow 0 \text{ as } n \rightarrow \infty \text{ because of weak convergence}} + \underbrace{\int_a^b (v_\infty - v_n)^2}_{\geq 0} \end{aligned}$$

What about the case

$$\int_a^b \psi(v_n)$$

Theorem Assume that $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is convex
Assume that $v_n \rightarrow v_\infty$ in $L^2((a,b))$

Then

$$\liminf_{n \rightarrow \infty} \int_a^b \psi(v_n) \geq \int_a^b \psi(v_\infty)$$

Proof. Let us cheat a little bit...

Due to convexity it turns out that

$$\psi(a+b) \geq \psi(a) + \psi'(a)b \quad \forall a \in \mathbb{R} \quad \forall b \in \mathbb{R}$$

Apply this inequality with $a = v_\infty(x)$, $b = v_n(x) - v_\infty(x)$:

$$\int_a^b \underbrace{\psi(v_n(x))}_{a+b} \geq \int_a^b \underbrace{\psi(v_\infty(x))}_a + \int_a^b \underbrace{\psi'(v_\infty(x))}_a \underbrace{(v_n(x) - v_\infty(x))}_b$$

Taking the limit

$$\lim_{n \rightarrow \infty} LHS = \int_a^b \psi(v_\infty(x)) + \lim_{n \rightarrow \infty} \underbrace{\int_a^b \psi'(v_\infty) \cdot (v_n - v_\infty)}_{\substack{\downarrow 0 \\ \text{because } v_n \rightarrow v_\infty}}$$

Remark ① We assumed that ψ is of class C^1
 ② We assumed that $\psi'(v_\infty) \in L^2((a,b))$.
 A sufficient condition is that ψ' has linear growth, namely $\exists A$ and B s.t.

$$|\psi'(x)| \leq A|x| + B \quad \forall x \in \mathbb{R}$$

$$|\psi'(v_\infty)|^2 \leq \underbrace{(A|v_\infty| + B)^2}_{\text{integrable if } v_\infty \in L^2}$$

What about NBC (it seems they do not make sense in H^1)

$$\min \left\{ \int_a^b \dot{u}^2 + \arctan u : u(a) = A \right\}$$

As before, we can apply the direct method in H^1

Main points : $\rightarrow$ DBC controls $|u_n(x_n)|$ with $x_n = a$
 $\rightarrow$ if $F(u_n) \leq M$, then

$$\|\dot{u}_n\|_2^2 \leq M - \int_a^b \arctan u_n \leq M + \frac{\pi}{2} (b-a)$$

and this controls oscillations

After existence in H^1 , we set $\psi(t) := F(u + t v)$ and we find that

$$\int_a^b 2\dot{u} \ddot{v} + \frac{1}{1+u^2} v$$

As before $\dot{u} \in H^1$ and $(2\dot{u})' = \frac{1}{1+u^2} = 2\ddot{u}$

Since we already know that $\dot{u} \in H^1$ (hence continuous) then the NBC $\dot{u}(b) = 0$ makes sense.

(NBC originates as usual after integration by parts if we consider v with $v(b) \neq 0$)

How to obtain a bound on $\|\dot{u}\|_2$?

$$F(u) = \int_a^b \varphi(\dot{u}) + \psi(x, u)$$

$$\text{Want : } F(u) \leq M \Rightarrow \|\dot{u}\|_2 \leq N$$

Answer : quadratic growth on φ and boundedness from below of ψ

$$\varphi(x) \geq Ax^2 - B \quad \text{for some } A > 0, B \in \mathbb{R}$$

$$\psi(x, s) \geq -C \quad \text{for some } C \in \mathbb{R}$$

$$M \geq F(u) \geq \int_a^b A \dot{u}^2 - B - C \quad \leadsto \text{conclusion}$$