

DIRECT METHOD

ROAD MAP

- ① Weak formulation : finding a suitable functional setting (Sobolev spaces)
- ② Compactness : finding a suitable notion of convergence wrt which sublevels are compact.
- ③ Lower semi-continuity : of the functional with respect to the previous notion of conv.
- ④ Regularity : back to classical solutions, usually solutions are more regular than expected in ①.

Example

$$F(u) = \int_0^1 [u^2 + \cos(u^3)] dx$$

Pbm : $\min \{ F(u) : u \in C^1([0,1]), u(0)=2, u(1)=7 \}$

- ① Weak formulation Extend the function to $H^1((0,1))$ because in this space

$$\int_0^1 u^2 dx$$

is well-defined and finite.

Remark If $u \in H^1((0,1))$, then u is continuous in $[0,1]$ so that pointwise conditions such as DBC make sense for u .

On the contrary, pointwise conds such as $\dot{u}(c) = C$ make no sense in H^1 because $\dot{u}$ is just L^2 .

② Compactness of sublevels. We need a notion of conv. s.t.

$\{u \in H^1 : u \text{ satisfies } \mathcal{B}c \text{ and } F(u) \leq M\}$
is compact.

A reasonable choice is the following: we say that $\{u_n\}$ converges to some u_∞ if

$$u_n \rightarrow u_\infty \text{ unif. in } [0,1] \quad u_n \rightarrow u_\infty \text{ weakly}$$

Remark If we ask that $\{u_n\}$ is bounded in $L^2([0,1])$, then the sequence $\{u_n\}$ is equicontinuous and therefore any convergence on u_n (for example $u_n \rightarrow u_\infty$ in L^2 or even $u_n \rightarrow u_\infty$ weakly) implies unif. conv.

Let us take a sublevel

$$\int_0^1 [u^2 + \underbrace{\cos(u^3)}_{\geq -1}] \leq M$$

$$\Rightarrow \int_0^1 u^2 dx \leq M - \int_0^1 \cos(u^3) dx \leq M+1$$

If we take any sequence $\{u_n\} \subseteq \text{sublevel}$, then

$$\|u_n\|_{L^2}^2 \leq M+1$$

and therefore there exists $u_{n_k} \rightarrow v_\infty$ weakly

In order to prove unif. conv. of u_{n_k} (up to further subseq.) we want to apply Ascoli - Arzelà. So we need to check

(i) equiboundedness

(ii) equicontinuity

Equicontinuity follows from the bound on derivatives

$$|u_m(x) - u_m(y)| \leq \underbrace{\| \dot{u}_m \|_{L^2}}_{\text{bounded}} \cdot |x-y|^{1/2} \quad \forall (x,y) \in [0,1]^2$$

(see lecture 14)

Equiboundedness follow from BC + equicontinuity

$$|u_m(x)| \leq \underbrace{|u_m(0)|}_{\substack{2 \\ \text{(B.C)}}} + \underbrace{|u_m(x) - u_m(0)|}_{\text{bounded because of previous estimate}}$$

A.A. $\Rightarrow u_{m_k}$ has a subseq. (not relabeled) such that

$$u_{m_k} \rightarrow u_\infty \text{ unif.}$$

We need that $v_\infty = \dot{u}_\infty$ (weak derivative) (for this it is enough to have weak conv. of functions and derivatives)

③ Lower semicontinuity Need to prove that

$$\left. \begin{array}{l} u_m \rightarrow u_\infty \text{ unif} \\ \dot{u}_m \rightarrow \dot{u}_\infty \end{array} \right\} \Rightarrow \liminf_{m \rightarrow \infty} F(u_m) \geq F(u_\infty)$$

Let us consider $\int_0^1 \cos(u_m^3) dx$

This converges to $\int_0^1 \cos(u_\infty^3) dx$ because

$$\cos(u_m^3) \rightarrow \cos(u_\infty^3) \text{ uniformly in } [0,1].$$

Let us consider $\int_0^1 \dot{u}_m^2 dx$. This is the square of the norm of $\dot{u}_m$ in $L^2((0,1))$, and the norm is LSC, and hence

$$\liminf_{n \rightarrow \infty} \int_0^1 u_n^2 \geq \int_0^1 u_\infty^2$$

In conclusion

$$\underbrace{\int_0^1 u_n^2}_{\substack{\text{limiting} \\ \text{with the} \\ \text{right ineq.}}} + \underbrace{\int_0^1 \cos(u_n^3)}_{\text{limit}}$$

Summary After ①, ②, ③ we know that the minimum problem admits at least solution in $H^1((0,1))$.

Remark The limit function u_∞ satisfies the BC because of unif. conv.

④ Regularity Let u_0 be a minimum point in H^1 .
Let v be any function in $C_c^\infty((0,1))$. We can compute the first variation in H^1 :

$$\psi(t) := F(u_0 + tv) := \int_0^1 (u_0 + tv)^2 + \cos(u_0 + tv)^3$$

$$\psi'(0) = \int_0^1 2u_0 \dot{v} - \sin(u_0)^3 \cdot 3(u_0)^2 v = 0$$

↑
this requires computing the derivative of an integral depending on a parameter.

$$\underbrace{\int_0^1 2u_0 \dot{v}}_f = \int_0^1 \underbrace{\sin(u_0^3) 3u_0^2 \cdot v}_{-g} \quad \forall v \in C_c^\infty((0,1))$$

This is equivalent to saying that f admits a weak derivative and this weak derivative is actually g .

Therefore $\dot{u}_0 \in H^1$, and hence $u_0 \in H^2$ and

$$2 \ddot{u}_0 = -3u(u^3)' - 3u^2$$

↑
weak second
order derivative

Diff. form of Euler equ.

BOOTSTRAP ARGUMENT

$u_0 \in H^2 \leadsto u_0$ is cont. $\leadsto$ RHS is cont. $\leadsto \ddot{u}_0$ is cont
 $\leadsto u_0 \in C^2 \leadsto \text{RHS} \in C^2 \leadsto \ddot{u}_0 \in C^2$
 $\leadsto u_0 \in C^4 \leadsto \dots$

and so on.

Conclusion: any minimum point of $F(u)$ in H^1 is actually of class C^∞ , and it is a classical solution of the usual Euler equ.

Summary: usually regularity has two steps

→ from H^1 to H^2 (here we exploit the defn. of weak derivative)

→ on H^2 we can write Euler equ. in diff. form

→ from H^2 to C^∞ (easy bootstrap argument)

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