

Sobolev functions are antiderivatives of their weak derivative

Let $f \in L^p(a,b)$, and let $g \in L^p(a,b)$ be the weak derivative
Let us set

$$G(x) := \int_a^x g(s) ds \quad \forall x \in (a,b)$$

Proposition $f(x) - G(x) = \text{constant}$ (up to a set of measure $= 0$)

Proof. $\int_a^b \varphi'(x) G(x) dx = \int_a^b \varphi'(x) \int_a^x g(s) ds dx$

$$= \int_a^b \int_a^x \varphi'(x) g(s) ds dx = \iint_T \varphi'(x) g(s) dx ds$$

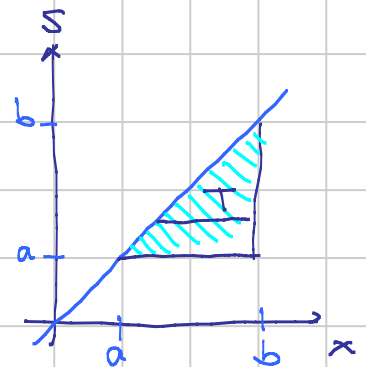
$$= \int_a^b ds \int_s^b dx \varphi'(x) g(s)$$

(Fubini
Tonelli)

$$= \int_a^b g(s) ds \int_s^b \varphi'(x) dx$$

$$= - \int_a^b g(s) \varphi(s) ds = - \int_a^b g(x) \varphi(x) dx$$

defn. of $\int_a^b f(x) \varphi'(x) dx$



Compare LHS and RHS + DBR lemma $\Rightarrow G - f$ is constant.

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Theorem $W^{1,p} \subseteq H^{1,p}$ both in (a,b) and in $\mathbb{R}$

Proof. Assume that $f \in W^{1,p}$ and g is the weak derivative.

We can approximate g with $C^\infty \ni g_n \rightarrow g$ in L^p

We can define f_n as antiderivatives of g_n and we have that

$$\begin{array}{ccc} g_n \rightarrow g & \text{in } L^p & f_n \rightarrow f \text{ in } L^p \\ \text{"} & & \\ f_n' & & \end{array}$$

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HÖLDER CONTINUITY OF SOBOLEV FUNCTIONS

Assume that $f \in W^{1,p}(a,b)$ and g is the weak derivative. Then f is $\frac{1}{q}$ -Hölder continuous, namely there exists M s.t.

$$|f(x_1) - f(x_2)| \leq M |x_1 - x_2|^{1/q} \quad \forall x_1, x_2 \in (a,b)$$

Proof. We know that $f(x)$ is "the" antiderivative of g , and hence

$$|f(x_2) - f(x_1)| = \left| \int_{x_1}^{x_2} g(s) ds \right|$$

$$\begin{array}{c} | \quad | \quad | \quad | \\ a \quad x_1 \quad x_2 \quad b \end{array}$$

$$\leq \int_{x_1}^{x_2} |g(s)| \cdot 1 \, ds$$

$\in L^p \quad \in L^q$

$$\frac{1}{p} + \frac{1}{q} = 1$$

(Hölder's ineq.) $\leq \left[\int_{x_1}^{x_2} |g(s)|^p ds \right]^{1/p} \cdot \left[\int_{x_1}^{x_2} 1^q ds \right]^{1/q}$

$$\leq \left[\int_a^b |g(s)|^p ds \right]^{1/p} |x_2 - x_1|^{1/q}$$

$$= \|g\|_{L^p(a,b)} \cdot |x_2 - x_1|^{1/q}$$

Result

$$|f(x_2) - f(x_1)| \leq \|g\|_{L^p([a,b])} \cdot |x_2 - x_1|^{1/q}$$

Special case If $p = q = 2$, we have that functions in $H^1([a,b])$ or $W^{1,2}([a,b])$ is $\frac{1}{2}$ -Hölder continuous.

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ASCOLI - (ARZELÀ) THEOREM

Let $f_n: [a,b] \rightarrow \mathbb{R}$. Let us assume that

(i) $\{f_n\}$ are EQUIBOUNDED, namely there exists $M \in \mathbb{R}$ s.t.

$$|f_n(x)| \leq M \quad \forall n \in \mathbb{N}, \forall x \in [a,b]$$

(ii) $\{f_n\}$ are EQUICONTINUOUS, namely

$$\forall \varepsilon > 0 \quad \forall x \in [a,b] \quad \exists \delta > 0 \text{ s.t.}$$

$$|f_n(x) - f_n(y)| \leq \varepsilon \quad \forall n \in \mathbb{N} \quad \forall y \in [x - \delta, x + \delta] \cap [a,b].$$

Then $\{f_n\}$ admits a subsequence converging uniformly, namely there exists $n_k \rightarrow +\infty$ and $f_\infty: [a,b] \rightarrow \mathbb{R}$ such that

$$f_{n_k} \rightarrow f_\infty \text{ uniformly.}$$

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More general statement : $f_n: X \rightarrow Y$, with X compact.
 $\uparrow \quad \uparrow$
metric spaces

Assumption (ii) is the same, and (i) becomes

(i-bis) $\forall x \in X \quad \exists K_x \subseteq Y$ compact such that

$$f_n(x) \in K_x \quad \forall n \in \mathbb{N} \quad \forall x \in X.$$

Remark Assume that $\{f_n\}$ are equi-Hölder continuous, namely

$$|f_n(x_2) - f_n(x_1)| \leq M \cdot |x_2 - x_1|^\alpha \quad \forall n \in \mathbb{N}, \forall x_1, x_2 \in \dots$$

with the same M and the same $\alpha \in (0, 1]$ for all n 's.

Then equicontinuity follows.

Let's go back to defn. (H)

$$f_n \rightarrow f \text{ in } L^p(a, b) \quad f'_n \rightarrow g \text{ in } L^p(a, b)$$

Use only that $\|f_n\|_{L^p(a, b)}$ is bounded. Use previous estimate

$$\begin{aligned} |f_n(x_2) - f_n(x_1)| &\leq \|f_n\|_{L^p(a, b)} |x_2 - x_1|^{1/q} \\ &\leq M |x_2 - x_1|^{1/q} \end{aligned}$$

This proves that the family $\{f_n\}$ is equicontinuous.

We need equiboundedness: $f_n \rightarrow f$ in $L^p(a, b)$, so again

$$\|f_n\|_{L^p(a, b)} \text{ is bounded, therefore } \int_a^b |f_n(x)|^p dx \leq M$$

" mean value theorem

$$(b-a) |f_n(x_n)|$$

$$\Rightarrow |f_n(x_n)| \leq M_1 \Rightarrow f_n \text{ is equibounded in some point } x_n.$$

For any $x \in [a, b]$ we have

$$\begin{aligned} |f_n(x)| &\leq |f_n(x_n)| + |f_n(x) - f_n(x_n)| \\ &\leq M_1 + M |x - x_n|^{1/q} \\ &\leq M_1 + M |b - a|^{1/q} \end{aligned}$$

In conclusion $\{f_n\}$ satisfies the assumptions of AA. theorem, and therefore (up to subsequences)

$$f_{n_k} \rightarrow f_0 \text{ uniformly and the limit is continuous}$$

↓
same limit as in L^p

At this point we know also that the whole sequence $f_n \rightarrow f$ uniformly.

Remark This provides alternative proof that Sobolev functions are Hölder cont.

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Assume $f \in W^{1,p}((a,b))$. Conjecture: $\sin f \in W^{1,p}((a,b))$
and $(\sin f)' = \cos f \cdot g$

"Exercise": prove it through definition (w)

Using (H) it's easier $f_n \rightarrow f \quad f_n' \rightarrow g$

$$\sin f_n \rightarrow \sin f \quad (\sin f_n)' = \cos f_n \cdot f_n' \rightarrow \cos f \cdot g$$

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Relation weak derivative — difference quotient

$$Q_R(x) = \frac{f(x+R) - f(x)}{R} \quad f: \mathbb{R} \rightarrow \mathbb{R}$$

Useful facts (exercises in the case $p=2$)

$$f \in W^{1,p} \Leftrightarrow Q_R(x) \rightarrow g \text{ in } L^p \Leftrightarrow Q_R(x) \rightarrow g \text{ in } L^p$$

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$\|Q_R\|_{L^p}$ equibounded
indep. of R .

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