

Calculus of Variations - LECTURE 13

Note Title

19/10/2016

SOBOLEV SPACES (In dimension 1)

Def. (W) let $f \in L^1(\mathbb{R})$ and let $g \in L^1(\mathbb{R})$.

We say that g is the weak derivative of f if

$$\int_{\mathbb{R}} f(x) \varphi'(x) dx = - \int_{\mathbb{R}} g(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(\mathbb{R})$$

We say that $f \in W^{1,p}(\mathbb{R})$ if $f \in L^p(\mathbb{R})$ and f has a weak derivative $g \in L^p(\mathbb{R})$.

Remark In the same way we can define k -th order derivatives and $W^{k,p}(\mathbb{R})$

$$f \in W^{k+1,p}(\mathbb{R}) \Leftrightarrow f \in W^{k,p}(\mathbb{R}) \text{ and the weak derivative } g \in W^{k,p}(\mathbb{R})$$

Remark In the same way we can define $W^{1,p}(a,b)$. In this case

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty(a,b)$$

PROPERTIES - EXAMPLES

① Uniqueness The weak derivative, when it exists, is unique.

Proof. Let g_1 and g_2 be two weak derivatives of f in (a,b) .
Then

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g_1(x) \varphi(x) dx = - \int_a^b g_2(x) \varphi(x) dx \quad \forall \varphi \in \dots$$

Then
$$\int_a^b [g_1(x) - g_2(x)] \varphi(x) dx = 0 \quad \forall \varphi \in C_c^\infty(a, b)$$

Conclusion by FLCV.

② $W^{1,p}(a, b)$ is a vector space and the weak derivative is a linear function.

Proof. Easy exercise

③ Consistency with classical notions

[3.1] If $f \in C^1(a, b)$, then $g = f'$ (integration by parts)

[3.2] If the weak derivative of f is some $g \in C^0(a, b)$, then actually f is of class C^1 , and coincides with some antiderivative of g .

Proof. Let $G(x)$ be any antiderivative of $g(x)$. Then

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g(x) \varphi(x) dx = \int_a^b G(x) \varphi'(x) dx$$

$\uparrow$ defn. of g
 $\uparrow$ int. by parts (classical)

$$\Rightarrow \int_a^b [f(x) - G(x)] \varphi'(x) dx = 0 \quad \forall \varphi \in C_c^\infty(a, b)$$

DBR Lemma $\Rightarrow f(x) - G(x)$ is constant.

④ $f(x) = |x|$ with $(a, b) = (-1, 1)$. Then it has a weak derivative which is

$$g(x) = \begin{cases} -1 & x \in (-1, 0) \\ 1 & x \in (0, 1) \end{cases}$$

Proof.

$$\begin{aligned}
 \int_{-1}^1 f(x) \varphi'(x) dx &= \int_{-1}^0 (-x) \varphi'(x) dx + \int_0^1 x \varphi'(x) dx \\
 &= \underbrace{\int_{-1}^0 1 \cdot \varphi(x) dx}_{\text{blue}} + \underbrace{[-x \cdot \varphi(x)]_{-1}^0 + [x \varphi(x)]_0^1 - \int_0^1 1 \cdot \varphi(x) dx}_{\text{blue}} \\
 &= - \int_{-1}^1 g(x) \varphi(x) dx
 \end{aligned}$$

In an analogous way, all functions which are continuous and $PW C^1$ do admit a weak derivative

Proof. Easy exercise.

⑤ $f(x) = \text{Heaviside function} = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

This function has no weak derivative $g(x)$ in $(-1, 1)$

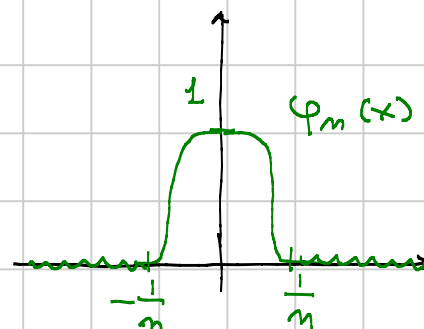
Proof. Let g be the weak derivative of f . Then

$$\int_{-1}^1 f(x) \varphi'(x) dx = - \int_{-1}^1 g(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty((-1, 1))$$

$$\int_0^1 \varphi'(x) dx = -\varphi(0)$$

$$\Rightarrow \int_{-1}^1 g(x) \varphi(x) dx = \varphi(0) \quad \forall \varphi \in \dots$$

$$\int_{-1}^1 g(x) \varphi_n(x) dx = \varphi_n(0) = 1 \quad \forall n \in \mathbb{N}$$



$\downarrow$ 0 $\downarrow$ 1
 pointwise limit + domination
 + dominated convergence theorem.

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Def. (H) We say that $f \in H^{1,p}((a,b))$ and $g \in L^p((a,b))$ is its weak derivative if there exists a sequence $\{f_n\} \subseteq C^\infty((a,b))$ (not asking compact support) such that

$$f_n \rightarrow f \text{ in } L^p((a,b)) \quad f_n' \rightarrow g \text{ in } L^p((a,b))$$

Same defn. in more abstract setting

Let us consider $C^1((a,b))$ or $C^\infty((a,b))$ with the norm

$$\|f\|_p := \|f\|_p + \|f'\|_p$$

The norm induces a structure of metric space. Then $H^{1,p}((a,b))$ is the completion of this metric space.
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Properties

① Uniqueness $\leadsto$ not so clear

② Vector space $\leadsto$ easy exercise

③ Consistency with classical case

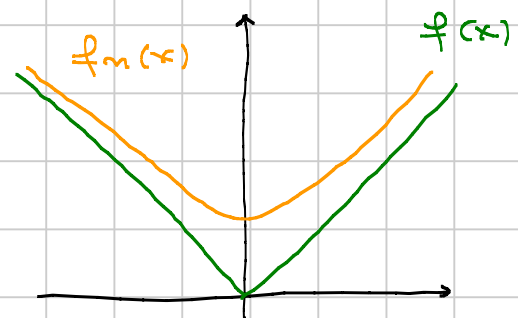
③.1 Easy if we admit $f_n \in C^1((a,b))$

③.2 Less easy

④ $f(x) = |x| \in H^{1,p}((-1,1))$

Can take $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$

Easy to check that $f_n \rightarrow f$ in $L^p((-1,1))$
 $f_n' \rightarrow -1/1$ in $L^p((-1,1))$



⑤ Heaviside function $\notin H^{1,p}((-1,1)) \rightsquigarrow$ not so clear.
 $\text{---} \circ \text{---} \circ \text{---}$

Theorem $f \in W$ (and g is the weak derivative)

Proof. We know that $f_n \rightarrow f$ and $f_n' \rightarrow g$ in $L^p((a,b))$
 For every $n \in \mathbb{N}$:

$$\int_a^b f_n(x) \varphi'(x) dx = - \int_a^b f_n'(x) \varphi(x) dx \quad \forall \varphi \in \dots$$

$$\int_a^b f(x) \varphi'(x) dx = - \int_a^b g(x) \varphi(x) dx$$

Both integrals converge because of dominated convergence.

Corollary All the missing points in the previous properties have now been proved.
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Theorem let us assume that $\{f_n\} \subseteq W^{1,2}((a,b))$ and let $\{g_n\} \subseteq L^2((a,b))$ be the sequence of weak derivatives. Let us assume that

$$\begin{array}{ccc} f_n \longrightarrow f_\infty & \text{WEAKLY in } L^2((a,b)) & \\ g_n \longrightarrow g_\infty & \text{" " "} & \end{array}$$

Then $f_\infty \in W^{1,2}((a,b))$ and g_∞ is the weak derivative of f_∞ .

Proof. $\int_a^b f_n(x) \varphi'(x) dx = - \int_a^b g_n(x) \varphi(x) dx \quad \forall \varphi \in C_c^\infty$

$$\int_a^b f_\infty(x) \varphi'(x) dx = - \int_a^b g_\infty(x) \varphi(x) dx$$

Notation: $H^k((a,b)) = H^{k,2}((a,b)) = W^{k,2}((a,b))$