

Some basic facts about Lebesgue measure and Lebesgue spaces.

Passing to the limit in integrals

① BEPPO LEVI (monotone convergence theorem)

$$f_n(x) \geq 0 \text{ with } f_{n+1}(x) \geq f_n(x) \quad \forall x \in (a,b) \quad \forall n \in \mathbb{N}$$

Then

$$\int_a^b \underbrace{\sup_{n \in \mathbb{N}} f_n(x)}_{\downarrow \lim_{n \rightarrow +\infty} f_n(x)} dx = \sup_n \int_a^b f_n(x) dx$$

② FATOU'S LEMMA $f_n(x) \geq 0 \quad \forall n \in \mathbb{N} \quad \forall x \in (a,b)$

Then

$$\int_a^b \liminf_{n \rightarrow +\infty} f_n(x) dx \leq \liminf_{n \rightarrow +\infty} \int_a^b f_n(x) dx$$

③ DOMINATED CONVERGENCE THEOREM (Lebesgue's theorem)

Let us assume that

DOMINATION

$$|f_n(x)| \leq g(x) \quad \forall x \in (a,b) \quad \forall n \in \mathbb{N}$$

where $\int_a^b g(x) dx < +\infty$

Let us assume that $f_n(x) \rightarrow f_\infty(x) \quad \forall x \in (a,b)$
(pointwise convergence)

Then

$$\int_a^b f_\infty(x) dx = \lim_{n \rightarrow +\infty} \int_a^b f_n(x) dx$$

LEBESGUE SPACES

Let $(a,b) \subseteq \mathbb{R}$ be an interval, and let $p \geq 1$.

The space $L^p(a,b)$ is the set of all functions $f: (a,b) \rightarrow \mathbb{R}$ such that

$$\int_a^b |f(x)|^p dx < +\infty$$

We set

$$\|f\|_p := \left[\int_a^b |f(x)|^p dx \right]^{1/p}$$

Main properties:

→ L^p is a vector space

→ $\|f\|_p$ is a norm

→ L^p is complete wrt the norm, and therefore it is a Banach space (this point requires Lebesgue's measure)

Special case $p=2$: $L^2(a,b)$ is a Hilbert space with inner product

$$\langle f, g \rangle := \int_a^b f(x) \cdot g(x) dx$$

HÖLDER'S INEQUALITY

$f \in L^p(a,b)$, $g \in L^q(a,b)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$, namely

$$\left| \int_a^b f(x) g(x) dx \right| \leq \left[\int_a^b |f(x)|^p dx \right]^{1/p} \cdot \left[\int_a^b |g(x)|^q dx \right]^{1/q}$$

Special case $p=q=2$: Hölder's ineq. is just Cauchy-Schwarz in L^2 .

More general statement: Let us assume that

$$\frac{1}{p_1} + \dots + \frac{1}{p_k} = \frac{1}{r}$$

and let $f_i \in L^{p_i}$ for every $i=1, \dots, k$.

Then $f_1 \cdot f_2 \cdot \dots \cdot f_k \in L^r$ and

$$\|f_1 \cdot \dots \cdot f_k\|_r \leq \|f_1\|_{p_1} \cdot \dots \cdot \|f_k\|_{p_k}.$$

Approximation results

For every $f \in L^p$, there exists a sequence $\{f_n\} \subseteq C^\infty(a,b)$ such that

$$f_n \rightarrow f \quad \text{in } L^p$$

In the same way we can approximate with piece-wise constant functions or with C^∞ functions satisfying a finite number of conditions in points of (a,b)

Fundamental lemma in the Calc Var (for L^1 functions)

Let $f \in L^1(a,b)$ be such that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in C_c^\infty(a,b)$$

Then $f(x) = 0$ for every $x \in (a,b)$ (up to a set of meas = 0)

Proof. Let us take a sequence $\{v_m\} \in C_c^\infty$ such that

$$v_m \rightarrow f \quad \text{in } L^1(a,b)$$

Then
$$\int_a^b f(x) v_m(x) dx = 0 \quad \forall m \in \mathbb{N}$$

$$\downarrow$$

$$\int_a^b f(x) f(x) dx = 0 \Rightarrow f(x) \equiv 0 \text{ up to a set of measure 0.}$$

Remark 1 In the approx result we can assume also that the convergence $f_m \rightarrow f$ is in L^p , but also pointwise and dominated. Therefore the convergence of the integrals in the previous proof follows from Lebesgue's theorem.

Remark 2 Since (a,b) is a bounded interval, then

$$L^p \supseteq L^q \quad \text{for every } p < q$$

So the FLCV is true in every L^p space.

Remark 3 DLR Lemma can be proved in the same way.
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Useful fact L^p is a separable space for every $p \geq 1$

Possible countable dense subsets

→ Polynomials with rational coefficients (requires Stone-Weierstrass)

→ linear combinations with rational coeff. of characteristic functions of intervals with rational endpoints.

Consequence: L^2 admits a (actually many) orthonormal systems.

Example: a orthonormal system in $L^2((0, \pi))$ is $\{\sin(nx)\}_{n \in \mathbb{N}}$

up to normalization constants.

(the only nontrivial part is that their span is dense).

The same is true with $\{\cos(nx)\}_{n \in \mathbb{N}}$

Weak convergence in $L^2(a, b)$ is $\forall w \in L^2$

$$f_n \rightarrow f \iff \int_a^b f_n(x) w(x) dx \rightarrow \int_a^b f(x) w(x) dx$$

Example 1 $f_n(x) = \sin(nx)$ weakly converges to 0 in $L^2((0, \pi))$

Proof. It is enough to check the relation for every w which is a characteristic function of a subinterval.

$$\int_0^\pi \sin(nx) \cdot 1_{(a,b)}(x) dx = \int_a^b \sin(nx) dx = \frac{1}{n} \cdot \text{something} \downarrow 0$$

Example 2 $(a, b) = (0, 1)$

$f_n(x) = \varphi(nx)$ = "greek functions"

$f_n(x) \rightarrow f_\infty(x) \equiv \frac{1}{2}$ (proof as before)

