

General setting: H is a separable Hilbert space

$u_m \rightarrow u_\infty$ strongly if $\|u_m - u_\infty\| \rightarrow 0$

$u_m \rightarrow u_\infty$ weakly if $\langle u_m, w \rangle \rightarrow \langle u_\infty, w \rangle \quad \forall w \in H$

Lemma Let u_m and w_m be two sequences in H .

Let us assume that

(i) $u_m \rightarrow u_\infty \in H$,

(ii) $w_m \rightarrow w_\infty \in H$,

(iii) w_m is bounded, namely there exists $M \in \mathbb{R}$ s.t.

$$\|w_m\| \leq M \quad \forall m \in \mathbb{N}$$

Then

$$\langle u_m, w_m \rangle \rightarrow \langle u_\infty, w_\infty \rangle.$$

Remark If $u_m \rightarrow u_\infty$ and $w_m \rightarrow w_\infty$, then it may happen that

$$\langle u_m, w_m \rangle \not\rightarrow \langle u_\infty, w_\infty \rangle$$

Example: $u_m = w_m = e_n \leftarrow$ Hilbert basis

$$1 = \langle u_m, w_m \rangle, \text{ but } u_m \rightarrow 0, w_m \rightarrow 0$$

Dim. $|\langle u_m, w_m \rangle - \langle u_\infty, w_\infty \rangle| \leq |\langle u_m, w_m \rangle - \langle u_\infty, w_m \rangle| + |\langle u_\infty, w_m \rangle - \langle u_\infty, w_\infty \rangle|$

$$|\langle u_m - u_\infty, w_m \rangle| \leq \|u_m - u_\infty\| \cdot \underbrace{\|w_m\|}_{\leq M} \leq M \|u_m - u_\infty\| \rightarrow 0$$

$\langle u_\infty, w_m \rangle \rightarrow \langle u_\infty, w_\infty \rangle$ because of the definition of weak convergence. $\square$

Question : if I want to prove that $U_m \rightarrow U_\infty$, can I prove that

$$\langle U_m, w \rangle \rightarrow \langle U_\infty, w \rangle \quad (*)$$

only for some special choices of w ?

Answer ^{If $\{U_m\}$ is bounded, then,}
It is enough to check (*) for every $w \in W \subseteq H$ with the property that

$$\overline{\text{Span}(W)} = H \quad \text{in the sense of the distance}$$

Proof. Step 1 If (*) is true for every $w \in W$, then it is true for every $w \in \text{Span}(W)$.

$$\begin{aligned} \langle U_m, c_1 w_1 + \dots + c_k w_k \rangle &= c_1 \langle U_m, w_1 \rangle + \dots + c_k \langle U_m, w_k \rangle \\ &\quad \downarrow \qquad \qquad \qquad \downarrow \\ &= c_1 \langle U_\infty, w_1 \rangle + \dots + c_k \langle U_\infty, w_k \rangle \\ &= \langle U_\infty, c_1 w_1 + \dots + c_k w_k \rangle \end{aligned}$$

Step 2 If (*) is true for every $w \in W$, then it is true for every $w \in \overline{W}$.

Let us take any $w \in \overline{W}$. Let us fix $\varepsilon > 0$. Then there exists $w_\varepsilon \in W$ such that

$$\|w - w_\varepsilon\| \leq \varepsilon.$$

Now

$$\begin{aligned} \underbrace{|\langle U_m, w \rangle - \langle U_\infty, w \rangle|}_{\text{LHS}} &\leq |\langle U_m, w \rangle - \langle U_m, w_\varepsilon \rangle| \quad (1) \\ &\quad + |\langle U_m, w_\varepsilon \rangle - \langle U_\infty, w_\varepsilon \rangle| \quad (2) \\ &\quad + |\langle U_\infty, w_\varepsilon \rangle - \langle U_\infty, w \rangle| \quad (3) \end{aligned}$$

$$(1) = |\langle U_m, w - w_\varepsilon \rangle| \leq \|U_m\| \cdot \|w - w_\varepsilon\| \leq M \cdot \varepsilon$$

$$(2) = \dots \text{ this tends to 0 because } w_\varepsilon \in W$$

$$\textcircled{3} = |\langle v_\infty, w - w_\varepsilon \rangle| \leq \|v_\infty\| \cdot \|w - w_\varepsilon\| \leq \|v_\infty\| \cdot \varepsilon$$

In conclusion

$$\text{LHS} \leq M\varepsilon + \|v_\infty\| \cdot \varepsilon + |\langle v_m, w_\varepsilon \rangle - \langle v_\infty, w_\varepsilon \rangle|$$

and therefore

$$0 \leq \limsup_{n \rightarrow \infty} \text{LHS} \leq M\varepsilon + \|v_\infty\| \cdot \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, then $\limsup \text{LHS} = 0$.

Corollary. If $\{v_n\}$ is a bounded sequence, then $v_n \rightarrow v_\infty$ if and only if

$$\langle v_n, e_k \rangle \rightarrow \langle v_\infty, e_k \rangle \quad \forall k \in \mathbb{N}.$$

(components converge to components)

Remark Boundedness is required (if not the result is false)

Example:

$$v_n = n e_n$$

It is obvious that for every FIXED $k \in \mathbb{N}$ it turns out

$$\langle v_n, e_k \rangle = 0 \quad \text{for every } n > k$$

and hence the weak limit is 0, if it exists.

On the other hand, $v_n \not\rightarrow 0$. Why?

We need to find $w \in H$ such that $\langle v_n, w \rangle \rightarrow 0$

Let us take $w = \sum_{n=1}^{\infty} \frac{1}{n} e_n$ $\sum \frac{1}{n^2} < +\infty \Rightarrow w$ is well defined

$$\langle v_n, w \rangle = 1 \quad \text{for every } n \in \mathbb{N}.$$

Theorem (true but not needed in the sequel).

Let us assume that $u_n \rightarrow u_\infty$.

Then $\{u_n\}$ is bounded.

Sketch of proof. We know that $\langle u_n, w \rangle \rightarrow \langle u_\infty, w \rangle$ for every $w \in H$, and therefore

$|\langle u_n, w \rangle|$ is bounded $\forall w \in H$.

$$H_k := \{w \in H : |\langle u_n, w \rangle| \leq k\}$$

Step 1 H_k is closed and $\bigcup_{k \in \mathbb{N}} H_k = H$ (very easy)

Step 2 Baire's lemma $\Rightarrow \exists k_0$ such that H_{k_0} has nonempty interior, namely it contains a ball.

Step 3 If H_{k_0} has nonempty interior, then it contains a ball with center in the origin

Step 4 Up to changing k_0 , one of the H_k contains the unit ball B with center in the origin, namely

$$\exists k \in \mathbb{N} \text{ s.t. } |\langle u_n, w \rangle| \leq k \quad \forall w \in B$$

Step 5 Let us choose $w = \frac{u_n}{\|u_n\|}$ and obtain that

$$\|u_n\| \leq k \quad \forall n \in \mathbb{N}$$

as required. $\square$

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Exercise Let H be a Hilbert space of infinite dimension.
Then H does not admit a countable algebraic basis.

Proof. Let $\{v_n\}$ be a countable algebraic basis. Then

$$H = \bigcup_{k \geq 1} H_k \quad \text{where} \quad H_k = \text{Span} \{v_1, \dots, v_k\}$$

↑
closed subset with empty interior

This is impossible because of Baire's lemma.

Corollary. The space $\mathbb{R}[x]$ cannot become a Hilbert space,
because it admits a countable algebraic basis.

Hopefully trivial comment Let $\{e_n\}$ be a Hilbert basis.
Let us assume that $v_n \rightarrow v_\infty$.

$$\|v_n\|^2 = \langle v_n, e_1 \rangle^2 + \langle v_n, e_2 \rangle^2 + \dots + \langle v_n, e_k \rangle^2 + \dots$$

$$\downarrow \qquad \qquad \qquad \downarrow$$
$$\langle v_\infty, e_1 \rangle^2 + \langle v_\infty, e_2 \rangle^2 + \dots + \langle v_\infty, e_k \rangle^2 + \dots$$

$$= \|v_\infty\|^2$$

WRONG because series and term-wise limits DO NOT COMMUTE

Conclusion:

- 1 - There are more weakly conv. seq. than strongly conv. seq.
- 2 - More conv. seq. $\Rightarrow$ easier to be compact (and indeed balls are weakly compact but not strongly compact.)
- 3 - More conv. seq. $\Rightarrow$ more diff. to have continuity (the norm is strongly continuous, but only weakly LSC)