

STRONG vs WEAK CONVERGENCE

Let H be a separable Hilbert space.

Def. We say that $u_n \rightarrow u_\infty$ **STRONGLY** if

$$\|u_n - u_\infty\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty$$

Good news: the norm, namely the function

$$H \ni u \rightarrow \|u\| \in \mathbb{R}$$

is continuous wrt strong convergence, namely

$$u_n \rightarrow u_\infty \quad \Rightarrow \quad \|u_n\| \rightarrow \|u_\infty\|$$

Proof. $|\|u_n\| - \|u_\infty\|| \leq \|u_n - u_\infty\|$ as a consequence of triangle inequality. $\square$

Bad news If $\dim(H) = +\infty$, the closed balls are NOT COMPACT.

Proof. Take any Hilbert basis $\{e_n\}$. This is a sequence with the property that

$$\text{dist}(e_i, e_j) = \sqrt{2} \quad \text{for every } i \neq j.$$

This forbids the existence of converging subseq. (no subseq. can be a Cauchy sequence). $\square$

Def. We say that u_m **WEAKLY** converges to u_∞ , and we write

$$u_m \longrightarrow u_\infty$$

$$\text{if } \lim_{m \rightarrow +\infty} \langle u_m, v \rangle = \langle u_\infty, v \rangle \quad \forall v \in H$$

Prop
$$\begin{array}{l} u_m \longrightarrow u_\infty \\ w_m \longrightarrow w_\infty \end{array} \Rightarrow \begin{array}{l} u_m + w_m \longrightarrow u_\infty + w_\infty \\ \lambda u_m \longrightarrow \lambda u_\infty \end{array} \quad \forall \lambda \in \mathbb{R}$$

Proof: exercise

Prop. If $u_m \longrightarrow u_\infty$, then $u_m \longrightarrow u_\infty$.

Proof Assume that $u_m \longrightarrow u_\infty$. Then

$$\langle u_m, v \rangle = \langle u_\infty, v \rangle + \underbrace{\langle u_m - u_\infty, v \rangle}_{\downarrow 0}$$

Indeed

$$|\langle u_m - u_\infty, v \rangle| \leq \underbrace{\|u_m - u_\infty\|}_{\downarrow 0} \cdot \underbrace{\|v\|}_{\uparrow \text{C.S.}}$$

If $\dim(H) = +\infty$, then weak convergence does not imply strong convergence

Proof. Let $\{e_n\}$ be any orthonormal basis. I claim that

$$e_n \longrightarrow 0$$

Indeed for every $v \in H$ it turns out that

$$\langle e_n, v \rangle = \underbrace{c_n}_{\uparrow \text{component of } v} \quad \text{and} \quad c_n \rightarrow 0 \quad \text{because} \quad \sum_{n=0}^{\infty} c_n^2 < +\infty$$

Let us apply the definition of weak convergence with $v =$ any element of an orth. system

$$v_m \longrightarrow v_\infty \Rightarrow \forall k \in \mathbb{N} \quad \langle v_m, e_k \rangle \longrightarrow \langle v_\infty, e_k \rangle$$

In other words, weak convergence implies convergence of components.

Actually the converse is true, namely weak convergence is somewhat equivalent to convergence of components.

$$\begin{aligned} \text{In } \mathbb{R}^n : v_m \longrightarrow v_\infty &\Leftrightarrow \text{dist}(v_\infty, v_m) \rightarrow 0 \\ &\Leftrightarrow \text{the components of } v_m \text{ converge to} \\ &\quad \text{the components of } v_\infty \end{aligned}$$

In infinite dimension the two notions are different

$$\begin{array}{ll} \text{dist}(v_\infty, v_m) \rightarrow 0 & \text{IS STRONG CONVERGENCE} \\ \text{conv. of components} & \text{IS WEAK CONVERGENCE.} \\ \text{---} \circ \text{---} \circ \text{---} & \end{array}$$

Very good news : in a separable Hilbert space closed balls are weakly compact

Not so bad news : the norm is no more continuous, but it is at least LSC w.r.t to weak conv.

COMPACTNESS

Let H be a separable Hilbert space

Let $\{v_m\} \subseteq H$ be a bounded sequence, namely

$$\|v_m\| \leq M \quad \forall m \in \mathbb{N}$$

Then there exists $m_k \rightarrow \infty$ and $v_\infty \in H$ such that

$$v_{m_k} \longrightarrow v_\infty$$

Proof. Let us fix an orthonormal system $\{e_k\}$.

STEP 1 Let us consider e_1 . We know that

$\langle v_n, e_1 \rangle$ is a bounded sequence

$$|\langle v_n, e_1 \rangle| \overset{\text{C.S.}}{\leq} \|v_n\| \cdot \overset{1}{\|e_1\|} \leq M$$

Up to subsequences

$$\langle v_{n_k}, e_1 \rangle \rightarrow c_1 \in \mathbb{R}$$

Let us consider e_2 . Up to further subsequences $\langle v_{n_k}, e_2 \rangle \rightarrow c_2 \in \mathbb{R}$

With a DIAGONAL PROCEDURE we obtain a subsequence v_{n_k} such that

$$\langle v_{n_k}, e_i \rangle \rightarrow c_i$$

$\forall i \in \mathbb{N}$. From now on we assume that

STEP 2 We would like to define

$$\langle v_n, e_i \rangle \rightarrow c_i$$

$$v_\infty := \sum_{i=0}^{\infty} c_i e_i$$

We need to show that the series is convergent, and this is equivalent to

$$\sum_{i=0}^{\infty} c_i^2 < +\infty \quad (\text{series of numbers})$$

which in turn is equivalent to proving that partial sums are equi-bounded. Let $k \in \mathbb{N}$ be fixed. Then

$$M \geq \|v_n\|^2 = \sum_{i=0}^{\infty} \langle v_n, e_i \rangle^2 \geq \sum_{i=0}^k \langle v_n, e_i \rangle^2 \downarrow \sum_{i=0}^k c_i^2$$

STEP 3 I claim that $\langle v_n, v \rangle \rightarrow \langle v_\infty, v \rangle$ for every $v \in H$.
 To begin with, we prove the result when v is a finite linear combination of elements of $\{e_n\}$.

Let

$$v = a_1 e_1 + \dots + a_k e_k$$

Then

$$\begin{aligned} \langle v_n, v \rangle &= \langle v_n, a_1 e_1 + \dots + a_k e_k \rangle \\ &= a_1 \langle v_n, e_1 \rangle + \dots + a_k \langle v_n, e_k \rangle \\ &\quad \downarrow \qquad \qquad \qquad \downarrow \\ &= a_1 \langle v_\infty, e_1 \rangle + \dots + a_k \langle v_\infty, e_k \rangle \\ &= \langle v_\infty, v \rangle \end{aligned}$$

STEP 4 We prove the result for every v . To this end, we fix $\varepsilon > 0$ and we fix $v_\varepsilon \in H$ such that

$$\|v - v_\varepsilon\| \leq \varepsilon \quad \text{and} \quad v_\varepsilon \text{ has a finite number of components.}$$

Now

$$\langle v_n, v \rangle = \boxed{\langle v_n, v_\varepsilon \rangle} + \boxed{\langle v_n, v - v_\varepsilon \rangle}$$

$\downarrow$ $\qquad \qquad \qquad \downarrow$
 $\langle v_\infty, v_\varepsilon \rangle$ $\qquad$ uniformly small

$$|\langle v_n, v - v_\varepsilon \rangle| \leq \underbrace{\|v_n\|}_{\leq M} \cdot \underbrace{\|v - v_\varepsilon\|}_{\leq \varepsilon} \leq M\varepsilon$$

More rigorously

$$\begin{aligned} |\langle v_n, v \rangle - \langle v_\infty, v \rangle| &= |\langle v_n, v_\varepsilon \rangle - \langle v_\infty, v_\varepsilon \rangle + \langle v_n, v - v_\varepsilon \rangle| \\ &\leq \underbrace{\|\langle v_n, v_\varepsilon \rangle - \langle v_\infty, v_\varepsilon \rangle\|}_{\leq \varepsilon \text{ if } n \text{ is large enough.}} + \underbrace{\|\langle v_n, v - v_\varepsilon \rangle\|}_{\leq M\varepsilon} \end{aligned}$$

Prop. If $u_n \rightarrow u_\infty$, then

$$\liminf_{n \rightarrow \infty} \|u_n\| \geq \|u_\infty\|$$

Proof.

$$\|u_n\|^2 = \|u_\infty + (u_n - u_\infty)\|^2$$

$$= \|u_\infty\|^2 + \|u_n - u_\infty\|^2 + 2 \langle u_n - u_\infty, u_\infty \rangle$$

$$\geq \|u_\infty\|^2 + 2 \langle u_n - u_\infty, u_\infty \rangle$$

$$= \|u_\infty\|^2 + 2 \langle u_n, u_\infty \rangle - 2 \langle u_\infty, u_\infty \rangle$$

$$\downarrow$$
$$2 \langle \cancel{u_\infty}, \cancel{u_\infty} \rangle - 2 \langle \cancel{u_\infty}, \cancel{u_\infty} \rangle$$

We proved that

$$\|u_n\|^2 \geq \|u_\infty\|^2 + 2 \langle u_n, u_\infty \rangle - 2 \langle u_\infty, u_\infty \rangle$$

Taking the liminf on both sides we conclude.

Exercise Assume $\{u_n\}$ and $\{w_n\}$ are two sequences.

What do we need to assume in order to conclude that

$$\langle u_n, w_n \rangle \rightarrow \langle u_\infty, w_\infty \rangle$$

Are two weak conv. enough? $u_n \rightarrow u_\infty, w_n \rightarrow w_\infty$

What about $u_n \rightarrow u_\infty$ and $w_n \rightarrow w_\infty$.