

DIRECT METHOD

Goal: proving that min exists. Then possibly find it.

Given: $f: X \rightarrow \mathbb{R}$ where X is a set.

We need some structure on X :

- set
 - set with a NOTION OF CONVERGENCE
 - topology
 - metric space
 - Banach space
 - Hilbert space
 - $\mathbb{R}^n$
-] vector structure

Def. A notion of conv. on a set X is "the list of all converging sequences with corresponding limits". More precisely

$$\text{Seq}(X) = \{\text{sequences on } X\} = \{f: \mathbb{N} \rightarrow X\}$$

A notion of convergence in X is a subset of

$$\text{Seq}(X) \times X$$

$$x_n \rightarrow x_\infty$$

We do not ask any special requirement (a sequence can have multiple limits, a subsequence may have a different limit, a constant sequence may have no limit).

Def. A subset $K \subseteq X$ is **COMPACT** wrt to a notion of convergence if every sequence $\{x_n\} \subseteq K$ has a converging subsequence, namely $\exists n_k \rightarrow +\infty$ and $x_\infty \in K$ s.t.

$$x_{n_k} \rightarrow x_\infty$$

Def. A function $f: X \rightarrow \mathbb{R}$ is **lower semi-continuous (LSC)** wrt the notion of convergence if for every $x_n \rightarrow x_\infty$ it turns out that

$$\liminf_{n \rightarrow +\infty} f(x_n) \geq f(x_\infty)$$

The function f is upper semi-continuous (USC) if

$$\limsup_{n \rightarrow +\infty} f(x_n) \leq f(x_\infty)$$

Remark

- If a notion of conv. contains many conv. seqs. then compactness is easy and LSC is difficult
- If it contains few conv. seqs., then the opposite.

THEOREM (Weierstrass) Let $f: X \rightarrow \mathbb{R}$. Let us assume that on X there is a notion of conv. s.t.

(i) X is compact,

(ii) f is lower semi-continuous.

Then there exists

$$\min \{f(x) : x \in X\}$$

Proof. For sure there exists $\inf \{f(x) : x \in X\} = I$.

By characterization of $\inf$, there exist a sequence $\{x_n\} \subseteq X$ such that

$$f(x_n) \rightarrow I \in \mathbb{R} \cup \{-\infty\}$$

Since X is compact, there exists a converging subseq.

$$x_{n_k} \rightarrow x_\infty$$

Due to LSC

subseq. of real numbers

$$I = \lim_{n \rightarrow \infty} f(x_n) \stackrel{\downarrow}{=} \lim_{k \rightarrow \infty} f(x_{n_k}) \\ = \liminf_{k \rightarrow \infty} f(x_{n_k})$$

$$(LSC) \rightarrow \geq f(x_\infty)$$

$$\begin{array}{l} I \text{ is the} \\ \text{infimum} \end{array} \rightarrow \geq I$$

Then we have equality, and therefore $f(x_\infty) = I$. Just in the end we know that $I \in \mathbb{R}$. $\square$

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Def. A function $f: X \rightarrow \mathbb{R}$ is COERCIVE if there exists a compact set $K \subseteq X$ such that

$$\inf \{f(x) : x \in X\} = \inf \{f(x) : x \in K\}$$

Corollary Assume that $f: X \rightarrow \mathbb{R}$ is LSC and coercive. Then the minimum exists.

$$\boxed{\text{Proof.}} \quad \inf_{x \in X} f(x) = \inf_{x \in K} f(x) = \min_{x \in K} f(x).$$

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Corollary Let $f: X \rightarrow \mathbb{R}$. Let us assume that there exists $M \in \mathbb{R}$ such that the SUBLEVEL $\{x \in X : f(x) \leq M\}$ is compact and nonempty. Assume also f is LSC. Then the min exists.

Proof. Previous corollary with $K = \text{sublevel}$. $\square$

HILBERT SPACES

A Hilbert space H is

- a vector space
- has an inner product (symmetric and positive, namely $\langle v, v \rangle > 0$ for every $v \neq 0$)
- it is complete wrt the distance induced by the inner product.

inner product $\rightsquigarrow$ norm $\rightsquigarrow$ distance $\rightsquigarrow$ notion of convergence

$$\|v\|^2 = \langle v, v \rangle \quad \text{dist}(v_1, v_2) := \|v_1 - v_2\|$$

$$v_n \rightarrow v_\infty \Leftrightarrow \text{dist}(v_n, v_\infty) \rightarrow 0.$$

ORTHONORMAL SYSTEM

A subset $\{e_n\} \subseteq H$ is an orth. system (AKA HILBERT BASIS) if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

and for every $v \in H$ there exists real numbers c_n s.t.

$$v = \sum_{n=0}^{\infty} c_n e_n$$

$\uparrow$
numbers, namely the components of v wrt $\{e_n\}$

Series in a Hilbert space

$$\sum_{n=0}^{\infty} v_n = S_\infty \text{ means that}$$

$$S_n \xrightarrow{\uparrow} S_\infty$$

limit in H

$$\text{where } S_n = v_0 + v_1 + \dots + v_n$$

$\uparrow$
sum in H

Prop. Let $\{e_n\}$ be an orth. system, and let $\{c_n\}$ be a sequence of real numbers.

Then

$$\underbrace{\sum_{n=0}^{\infty} c_n e_n}_{\text{series of vectors}} \text{ converges} \iff \underbrace{\sum_{n=0}^{\infty} c_n^2}_{\text{series of numbers}} \text{ converges}$$

Proof. Let $S_m := c_0 e_0 + \dots + c_m e_m$ $\hat{S}_m := c_0^2 + \dots + c_m^2$

For every $m > n$:

$$\begin{aligned} \|S_m - S_n\|^2 &= \|c_{n+1} e_{n+1} + \dots + c_m e_m\|^2 \\ &= \|c_{n+1}\|^2 + \dots + \|c_m\|^2 \\ &= \hat{S}_m - \hat{S}_n \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} c_n^2 \text{ converges} \iff \{\hat{S}_n\} \text{ is a Cauchy sequence of numbers}$$

$$\iff \{S_n\} \text{ is a Cauchy seq. of vectors}$$

$$\iff \underbrace{\sum_{n=0}^{\infty} c_n e_n}_{\text{--- } 0 \text{ --- } 0 \text{ ---}} \text{ converges.}$$

Def. A Hilbert space H is SEPARABLE if there exists a countable dense subset.

Theorem. Let H be a separable Hilbert space.

Then H admits a (finite or countable) orthonormal system.

Importance of orthonormal systems: the same of the algebraic basis in finite dimension.

In finite dimension, we can identify any vector $v \in V$ with the set of its components.

Every finite dim. vector space V is $\mathbb{R}^n$

In every separable Hilbert space, we can identify each v with the countable set of its components.

In some sense, the space " $\mathbb{R}^\infty$ ".

In particular

$$\|v\|^2 = \sum_{n=0}^{\infty} c_n^2 \quad \text{where} \quad c_n = \langle v, e_n \rangle$$

In other words

$$v = \sum_{n=0}^{\infty} \langle v, e_n \rangle e_n.$$

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