

Example 1 (Image processing) Let $f: [a, b] \rightarrow \mathbb{R}$ be given (continuous function)

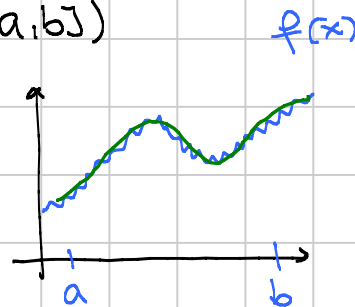
$$F(u) = \int_a^b u^2 + c(u-f)^2 dx$$

↑
positive parameter

Problem: $\min F(u)$ among all $u \in C^1([a, b])$

$p(x) \rightsquigarrow$ NOISY SIGNAL RECEIVED

argmin $\leadsto$ regularized version of f .



$$L(x, s, p) = p^2 + (s - f(x))^2$$

$$(2\dot{u})' = 2(u - f) \quad \begin{cases} \ddot{u} = u - f \\ \dot{u}(a) = 0 \\ \dot{u}(b) = 0 \end{cases} \leftarrow \text{linear ODE}$$

For ODEs theory $\leadsto$ existence and uniqueness of solution of class C^2

Since L is convex in (s, p) for every $x \in [a, b]$, the solution is the unique min. point of $F(u)$.

Qualitative properties of the curve m :

(1) $\int_a^b u = \int_a^b p$

Integrate ELE : $\int_a^b \ddot{u} = \int_a^b a - \int_a^b p$

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$\dot{u}(b) - \dot{u}(a) = 0$

$$\textcircled{2} \quad \begin{aligned} \max_{x \in [a,b]} u(x) &\leq \max_{x \in [a,b]} f(x) \\ \min u(x) &\geq \min f(x) \end{aligned}$$

Proof 1 Look at ELE $\ddot{u} = u - f$

Assume that the maximum point of u is some $x_0 \in (a, b)$

Then

$$0 \geq \ddot{u}(x_0) = u(x_0) - f(x_0)$$

$\Rightarrow u(x_0) \leq f(x_0) \leq \max_{x \in [a, b]} f(x)$. Same for the min.

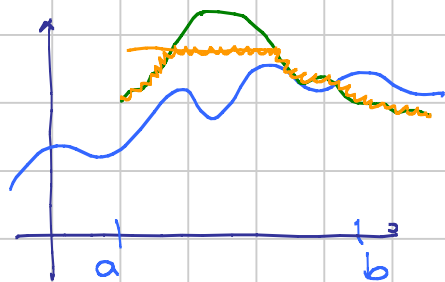
Proof 2 Purely variational proof: TRUNCATION ARGUMENT.

Assume $\max u > \max f$.

I can modify u by truncating with a constant in between.

In this way

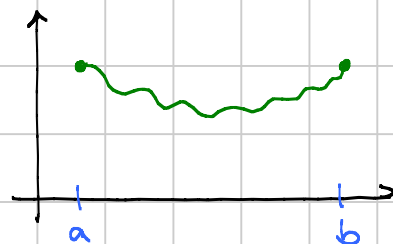
- we save in the term $\int \dot{u}^2$
- we save in the term $\int (u - f)^2$



Example 2 Equilibrium of a string

$$\min \left\{ \int_a^b \dot{u}^2 + k u : u(a) = A, u(b) = B \right\}$$

$\uparrow$
gravity



$$L(x, s, p) = p^2 + k s$$

$$(2\dot{u})' = k$$

$$2\ddot{u} = k + Bc$$

Solution: parabola fitting the boundary conditions.

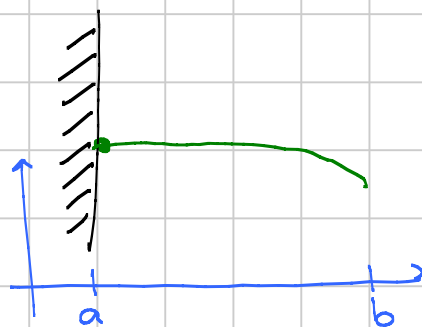
Example 3 Equilibrium of a BEAM (TRAVE in Italian)

$$\min \left\{ \int_a^b \ddot{u}^2 + k u : u(a) = A, \dot{u}(a) = 0 \right\}$$

$$\psi(t) := \int_a^b (\ddot{u} + t\ddot{v})^2 + k(u + tv)$$

$$\psi'(t) = \int_a^b 2\ddot{u}\ddot{v} + kv$$

1st integral form



$$\begin{aligned}
&= - \int_a^b 2 \ddot{u} \dot{v} + [\dot{v} \ddot{u}]_a^b + \int_a^b kv \\
&= \int_a^b 2 u^{(4)} v - [\ddot{u} \dot{v}]_a^b + [\dot{v} \ddot{u}]_a^b + \int_a^b kv
\end{aligned}$$

$$\begin{cases} 2u^{(4)} = k \\ u(a) = A \\ \dot{u}(a) = 0 \\ \ddot{u}(b) = 0 \\ \dddot{u}(b) = 0 \end{cases} \quad \begin{array}{l} \text{ELE} \quad \rightsquigarrow u \text{ is pol. of degree 4.} \\ \text{Imposed at the beginning} \\ \text{generated by integr. by parts.} \end{array}$$

Example 6 $F(u) = \int_a^b (\dot{u}^2 + u^2) dx$

$$\min \{ F(u) : u(a) = A, \dot{u}(b) = 1 \}$$

$$\begin{aligned}
&\ddot{u} = u \quad \text{ELE} \\
&u(a) = A \\
&\dot{u}(b) = 1 \\
&\ddot{u}(b) = 0 \quad \text{NOT COMPATIBLE}
\end{aligned}$$

↑ Appears as always

What happens: $\rightarrow$ min does NOT exist

$\rightarrow$ $\inf = \min$ without condition $\dot{u}(b) = 1$
(which exists because of usual convexity argument)

$\rightarrow$ Take $u_0(x)$ which realizes the minimum

$\rightarrow$ Modify it close to b in order to fulfil $\dot{u}_\varepsilon(b) = 1$.



The same if we add condition $u^{(2016)}(b) = 2017$

The problem has no solution UNLESS u_0 has that property

General fact: functionals with Lagrangians depending on derivatives up to order k ignore BC involving derivatives of order $\geq k$

Example 5 $\int_a^b \dot{u}^4 + u^2$ minimize with DBC

$$(4\dot{u}^3)' = 2u$$

$$12\dot{u}^2 \ddot{u} = 2u \quad \leadsto \text{difficult?}$$

ERDMANN form

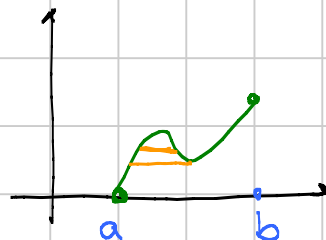
$$\dot{u} L_p(u, \dot{u}) = c + L(u, \dot{u})$$

$$4\dot{u}^4 = c + \dot{u}^4 + u^2$$

$$3\dot{u}^4 = c + u^2$$

$$\dot{u} = \sqrt[4]{c + \frac{1}{3}u^2}$$

Let us assume that $B > A = 0$



It is reasonable that $u \geq 0$ and $\dot{u} \geq 0$, so

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$$\dot{u} = \sqrt[4]{c + \frac{1}{3}u^2}$$

$$\int_a^b \frac{\dot{u}(x)}{\sqrt[4]{c + \frac{1}{3}u(x)^2}} = \int_a^b 1 = b - a$$

$$\parallel y = u(x) \quad dy = \dot{u}(x) dx$$

$$\begin{aligned} B &= u(b) \\ 0 &= u(a) \end{aligned} \quad \int \frac{dy}{\sqrt[4]{c + \frac{1}{3}y^2}} = b - a$$

Easy calculus: there is at least (actually exactly) one value of c s.t. $\int = b - a$

The corresponding solution u is the solution to the min. pbm. (Convexity).

Remark There was NO NEED to show $u \geq 0$, $\dot{u} \geq 0$ BEFORE.

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Summary of the INDIRECT METHOD

- ① Derive necessary conditions (ELE and BC)
- ② Hope to be able to solve ELE + BC (or to prove existence of a solution)
- ③ Hope to be able to conclude through
 - convexity
 - trivial lemma.

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