

Point-to-curve probs

$$F(u) = \int_a^b L(x, u, \dot{u}) dx$$

Given a function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ of class C^1

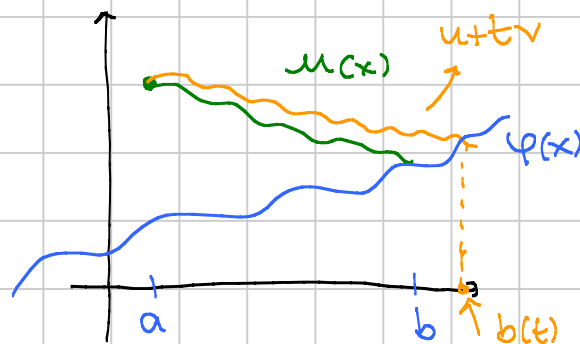
$$F(u, b) = \int_a^b L(x, u, \dot{u}) dx$$

Goal: minimizing $F(u, b)$ among all pairs (u, b) with

$$u \in C^1([a, b])$$

$$u(a) = A \leftarrow \text{given}$$

$$u(b) = \varphi(b)$$



As always, let us take any $v \in C^\infty([a, +\infty))$ with $v(a) = 0$. Consider

$$u(x) + t v(x)$$

Find whether and where the new function crosses $\varphi(x)$.

$$u(b(t)) + t v(b(t)) = \varphi(b(t))$$

$$b: (-\delta, \delta) \rightarrow \mathbb{R}$$

$$b(0) = b$$

$$u(b(t)) + t v(b(t)) - \varphi(b(t)) = 0$$

$$\Phi(x, t) := u(x) + t v(x) - \varphi(x)$$

$$\Phi(b(t), t) = 0$$

$\leadsto$ implicit function theorem $\leadsto$ can find $b(t)$ if

$$\Phi_x(b, 0) \neq 0$$

$$\Phi_x(x, t) = \dot{u}(x) + t \dot{v}(x) - \dot{\varphi}(x)$$

$$\Phi_x(b, 0) \neq 0 \Leftrightarrow \dot{u}(b) - \dot{\varphi}(b) \neq 0$$

Let us assume that $\ddot{u}(b) - \ddot{\varphi}(b) \neq 0$. Then $b(t)$ exists and

$$\dot{b}(0) = - \frac{\Phi_t(b, 0)}{\Phi_x(b, 0)} = - \frac{v(b)}{\ddot{u}(b) - \ddot{\varphi}(b)}$$

Let us set

$$\psi(t) := \int_a^{b(t)} L(x, u(x) + tv(x), \dot{u}(x) + t\dot{v}(x)) dx$$

$$\begin{aligned} \psi'(t) &= \dot{b}(t) L(b(t), u(b(t)) + tv(b(t)), \dot{u}(b(t)) + t\dot{v}(b(t))) \\ &\quad + \int_a^{b(t)} L_s(x, u + tv, \dot{u} + t\dot{v}) v + L_p(x, u + tv, \dot{u} + t\dot{v}) \dot{v} dx \end{aligned}$$

Setting $t=0$:

$$\begin{aligned} 0 = \psi'(0) &= \dot{b}(0) L(b, u(b), \dot{u}(b)) + \int_a^b L_s(x, u, \dot{u}) v + L_p(x, u, \dot{u}) \dot{v} \\ &= \int_a^b L_s(\dots) v - [L_p(\dots)]' v \\ &\quad + L_p(b, u(b), \dot{u}(b)) v(b) \end{aligned}$$

$$0 = \psi'(0) = - \frac{v(b)}{\ddot{u}(b) - \ddot{\varphi}(b)} L(b, u(b), \dot{u}(b)) + L_p(b, u(b), \dot{u}(b)) v(b)$$

Since we can take $v(b) = 1$, we obtain

$$L_p(b, u(b), \dot{u}(b)) [\ddot{u}(b) - \ddot{\varphi}(b)] = L(b, u(b), \dot{u}(b))$$

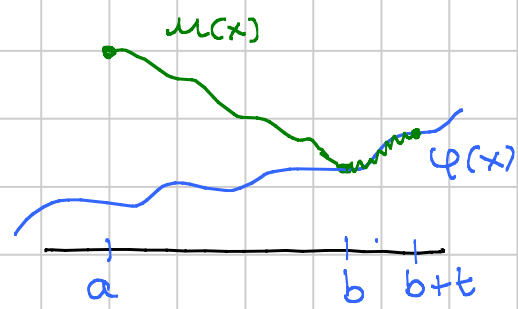
Boundary condition for point-to-curve problem
(TRANSVERSALITY)

In addition we have the usual ELE

$$[L_p(x, u(x), \dot{u}(x))] = L_s(x, u(x), \dot{u}(x)) \quad \forall x \in [a, b]$$

The transversality condition holds true even if $\dot{u}(b) = \dot{\varphi}(b)$

1st step Consider an "extension" of u up to $b+t$ obtained by following $\varphi(x)$



$$\psi(t) := \int_a^b L(x, u, \dot{u}) dx + \int_b^{b+t} L(x, \varphi, \dot{\varphi}) dx$$

$$\psi'(t) = L(b+t, \varphi(b+t), \dot{\varphi}(b+t))$$

$$\psi'(0) = L(b, \varphi(b), \dot{\varphi}(b)) = L(b, u(b), \dot{u}(b))$$

↑ because $\dot{u}(b) = \dot{\varphi}(b)$

If I could say that $\psi'(0) = 0$, then I could conclude that

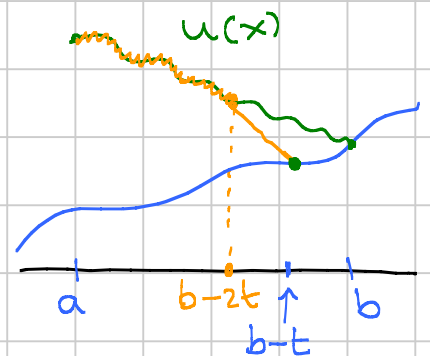
$$L(b, u(b), \dot{u}(b)) = 0$$

ψ is defined for $t \geq 0$ and has a minimum for $t=0$.
This is enough to conclude that $L(\dots) \geq 0$

2nd step Want to find a new function

$u(t, x)$ such that

$$u(t, b-t) = \varphi(b-t)$$



$$u(t, x) := \begin{cases} u(x) & x \in [a, b-2t] \\ \ell(x) & x \in [b-2t, b-t] \\ \text{affine interp} & x \in [b-2t, b-t] \end{cases}$$

$$\psi(t) := \int_a^{b-2t} L(x, u, \dot{u}) dx + \int_{b-2t}^{b-t} L(x, \ell(x), \dot{\ell}(x)) dx$$

Compute $\psi'(0) \geq 0$ and obtain... $L(\dots) \leq 0$

In order to compute the derivative of

$$\int_{b-2t}^{b-t} L(x, \ell(x), \dot{\ell}(x)) dx$$

DO DEPEND ON t

in $t=0$, it is simpler to use the definition

$$t L(x(t), \ell(x(t)), \dot{\ell}(x(t)))$$

Divide by t and take the limit as $t \rightarrow 0^+$

$\downarrow$
 b

$\downarrow$
 $u(b)$

$\downarrow$
 $\dot{u}(b)$

we need that

$$\ddot{u}(b) = \ddot{\varphi}(b)$$

— 0 — 0 —

Remark The modified function is just piecewise smooth. Let us assume that $u(x)$ is a minimum point in the PWC^1 class.

— 0 — 0 —

Detail :

$$\dot{\ell}(x) = \frac{\varphi(b-t) - u(b-2t)}{t}$$

$$= \frac{\varphi(b-t) - \varphi(b) + u(b) - u(b-2t)}{t}$$

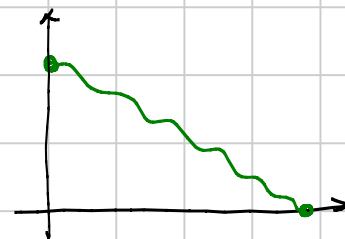
$$\downarrow$$

$$-\dot{\varphi}(b) + 2\dot{u}(b) = \dot{u}(b)$$

Example $F(u, b) = \int_a^b \dot{u}^2 dx$ $u(0) = \tau$, $\varphi(x) \equiv 0$

Transversality: $L_p(\dot{u} - \dot{\varphi}) = L$

$$2\dot{u}(b)(\dot{u}(b) - 0) = \dot{u}(b)^2 \Rightarrow \dot{u}(b) = 0$$



Euler equation: $(L_p)' = L_s$ $(2\dot{u})' = 0$ $\dot{u} = \text{const}$

$\Rightarrow \dot{u} \equiv 0 \Rightarrow u$ is constant $\Rightarrow$ impossible $\Rightarrow$ NO min in C^1

Actually, the infimum is 0 and a min. seq. is

$$F(u_n(x), n) = \int_0^n \left(\frac{x}{n}\right)^2 dx \rightarrow 0$$



— 0 — 0 —