

Calculus of Variations — LECTURE 06

Note Title

05/10/2016

Question: how to prove that a given function $u_0(x)$ is a/the minimum point of a functional $F(u)$

Basic fact about convex functions

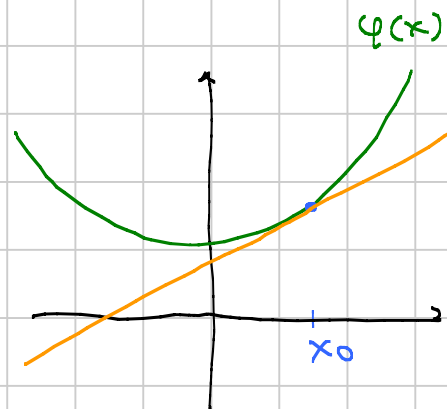
Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a convex function.

Let $x_0 \in \mathbb{R}$.

Then there exists $m \in \mathbb{R}$ such that

$$\varphi(x) \geq \underbrace{\varphi(x_0) + m(x - x_0)}_{\text{line through } (x_0, \varphi(x_0))}$$

strict inequality when φ is strictly convex and $x \neq x_0$



Proof when φ is of class C^2

Use Taylor with Lagrange remainder

Define $x - x_0 = R$. Then

$$\begin{aligned} \varphi(x) &= \varphi(x_0 + R) = \varphi(x_0) + \varphi'(x_0)R + \frac{1}{2} \varphi''(x_0 + \theta R) R^2 \\ &\quad \uparrow \text{Taylor-Lagrange} \\ &\geq \varphi(x_0) + \underbrace{\varphi'(x_0)}_{\substack{\text{min} \\ -0 \quad -0 \quad -}} (x - x_0) \end{aligned}$$

The result is true even if φ is not of class C^1

[Proof. In every $x_0 \in \mathbb{R}$ one can define

$$\varphi'_+(x_0) = \lim_{R \rightarrow 0^+} \frac{\varphi(x_0 + R) - \varphi(x_0)}{R}$$

$$\varphi'_-(x_0) = \lim_{R \rightarrow 0^-} \frac{\varphi(x_0 - R) - \varphi(x_0)}{-R}$$



For every $x_0 \in \mathbb{R}$ it turns out that $\varphi'_-(x_0) \leq \varphi'_+(x_0)$ and any m in between does the job.

The same is true in 2D or any dimension. For example in 2D if $f(x, y)$ is a convex function, then for every $(x_0, y_0) \in \mathbb{R}^2$ there exists a plane such that

$$f(x, y) \geq f(x_0, y_0) + m_1(x - x_0) + m_2(y - y_0)$$

with strict inequality if ---

MINIMALITY THROUGH CONVEXITY

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx \quad L(x, s, p)$$

Want to minimize F in $C^1([a, b])$ subject to BC
 $u(a) = A, \quad u(b) = B$

Let us assume that $u_0(x)$ is a solution to ELE with BC.

If $(s, p) \rightarrow L(x, s, p)$ is convex wrt (s, p) for every $x \in [a, b]$

Then $u_0(x)$ is a minimum point for $F(u)$

If the same function is strictly convex, then $u_0(x)$ is the unique minimum point.

Proof. Let us take any $w \in C^1$ with same BC.

Can write $w = u_0 + v$

$\uparrow$ class C^1 with zero BC

$$F(w) = F(u_0 + v) = \int_a^b L(x, u_0 + v, u_0' + v') dx$$

$$L \text{ is convex} \rightarrow \geq \int_a^b L(x, u_0, u_0') dx \quad \leftarrow F(u_0)$$

$$+ \underbrace{\int_a^b L_s(x, u_0, u_0') v + L_p(x, u_0, u_0') v'}_{= 0 \text{ because of ELE}}$$

This proves that $F(w) \geq F(u_0)$ for every admissible w .

Let us assume now that L is strictly convex and $\dot{v} \neq 0$ or $v \neq 0$.
Then there exists at least one point $x_0 \in [a, b]$ where

$$L(x_0, u_0(x_0) + v(x_0), \dot{u}_0(x_0) + \dot{v}(x_0)) > L(x_0, u_0, \dot{u}_0) + L_s v + L_p \dot{v}$$

The same is true in a neighborhood (because of cont.), and therefore $F(w) > F(u_0)$

Basic example

$$F(u) = \int_a^b L(\dot{u}) dx$$

$$L(p)$$

$$\text{ELE} \quad [L_p(\cdot)]' = L_s \quad [L_p(\dot{u})]' = 0$$

$$L_{pp}(\dot{u}) \ddot{u} = 0$$

If $L_{pp}(p) > 0$ for every $p \in \mathbb{R}$, then $\ddot{u} = 0$, which means that the only solution is the line. This is also the unique minimizer.

More generally let us consider $\min \left\{ \int_a^b L(\dot{u}) : \text{DBC} \right\}$

(1) If L is convex (no regularity required), then the line is a minimizer

(2) If L is strictly convex, then it is the unique minimizer.

Proof. Let $u_0(x) = \alpha x + \beta$ be the line. Let w be any other admissible function. Write $w = u_0 + v$. Take m such that

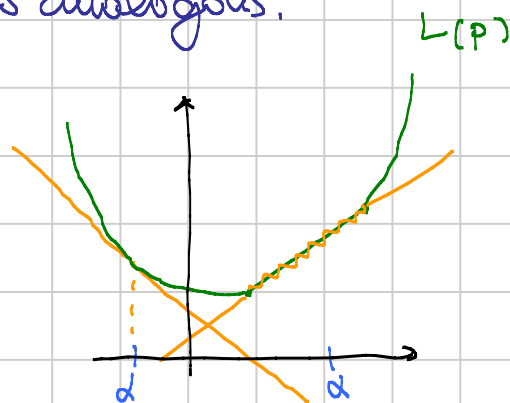
$$L(p) \geq L(\alpha) + m(p - \alpha)$$

As before

$$\begin{aligned}
 F(w) &= \int_a^b L(\dot{u}_0 + \dot{v}) = \int_a^b L(\alpha + \dot{v}) \\
 &\geq \int_a^b L(\alpha) + u \dot{v} \\
 &= F(u_0) + u(v(b) - v(a)) = F(u_0)
 \end{aligned}$$

The argument for the strictly convex case is analogous.

Remark If L is convex (but not necessarily strictly convex) and α is such that $L(p)$ is always above the tangent line, then $u_0(x)$ is again the unique minimizer.



Second tool in order to prove minimality

TRIVIAL LEMMA Let X be a set. Let $x_0 \in X$.
Let $F: X \rightarrow \mathbb{R}$ and $G: X \rightarrow \mathbb{R}$

Let us assume that

(i) $F(x) \geq G(x) \quad \forall x \in X$

(ii) $F(x_0) = G(x_0)$

(iii) $G(x) \geq G(x_0) \quad \forall x \in X$ (namely $x_0 \in \operatorname{argmin} \{G(x) : x \in X\}$)

Then

$$x_0 \in \operatorname{argmin} \{F(x) : x \in X\}.$$

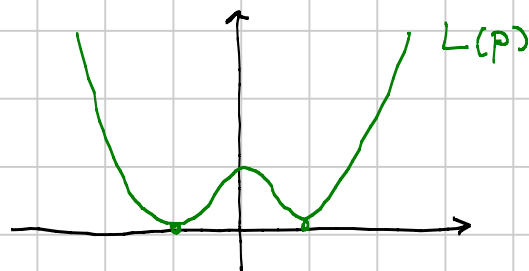
Proof. $F(x) \geq G(x) \geq G(x_0) = F(x_0) \quad \forall x \in X. \quad \square$

$\uparrow$ (i) $\uparrow$ (iii) $\uparrow$ (ii)

Example

$$\min \left\{ \int_0^1 \underbrace{(\dot{u}^2 - 1)^2}_{L(\dot{u})} dx : u \in C^1, u(0)=2, u(1)=4 \right\}$$

$$L(p) = (p^2 - 1)^2$$



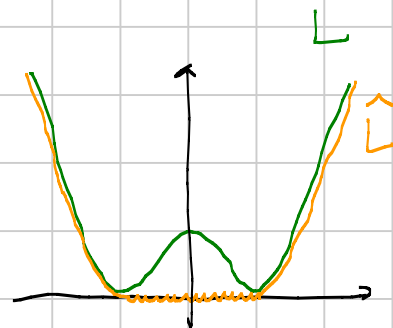
DOUBLE WELL POTENTIAL

Conjecture: the line $u_0(x) = 2 + 2x$ is the unique minimizer.

Let $F(u)$ be the given functional, and let

$$G(u) := \int_0^1 \hat{L}(\dot{u}) dx$$

where $\hat{L}(p)$ is the convexification of $L(p)$



$$F(u) \geq G(u) \quad (\text{because } L(p) \geq \hat{L}(p) \quad \forall p \in \mathbb{R})$$

$$F(u_0) = G(u_0) \quad (\text{because } L(2) = \hat{L}(2))$$

u_0 is a minimum point of $G(u)$ (because $\hat{L}$ is convex)

Trivial lemma $\Rightarrow u_0(x) = 2x + 2$ is a minimum point of F

Moreover, it is the unique minimum point of F because
" " " " " " " " G , because
 $p = 2$ falls in a region where $\hat{L}$ is strictly convex.