

DIFFERENT FORMS OF EULER Equation

(AKA EULER-LAGRANGE Eqn)

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx$$

↑ Lagrangian

$$L: [a, b] \times \mathbb{R} \times \mathbb{R}$$

Let us assume that u is a minimum point of F subject to some BC, for example DBC

$$u(a) = A, \quad u(b) = B$$

Then $0 = \delta F(u, v) = \lim_{t \rightarrow 0} \frac{F(u+tv) - F(u)}{t} = \psi'(0)$

$$\psi(t) := F(u+tv).$$

General result Let $f: \underbrace{[a, b]}_x \times \underbrace{[-\delta, \delta]}_t \rightarrow \mathbb{R}$

Let us set

$$\psi(t) := \int_a^b f(x, t) dx \quad t \in [-\delta, \delta]$$

- (1) If $f(x, t)$ is continuous in $[a, b] \times [-\delta, \delta]$, then $\psi(t)$ is continuous in $[-\delta, \delta]$
- (2) If $f_t(x, t)$ is continuous in $[a, b] \times [-\delta, \delta]$, then $\psi'(t)$ exists and it is given by

$$\psi'(t) = \int_a^b f_t(x, t) dx$$

(the derivative of the integral is the integral of the derivative)

Let us take

$$\psi(t) = F(u+tv) = \int_a^b \overbrace{L(x, u(x)+tv(x), \dot{u}(x)+t\dot{v}(x))}^{f(x,t)} dx$$

$$\psi'(t) = \int_a^b \frac{d}{dt} L(x, u+tv, \dot{u}+t\dot{v}) dx$$

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RULE

$$= \int_a^b L_s(x, u+tv, \dot{u}+t\dot{v})v + L_p(x, u+tv, \dot{u}+t\dot{v})\dot{v} dx$$

When $t=0$ we get

$$\psi'(0) = \delta F(u, v) = \int_a^b L_s(x, u(x), \dot{u}(x))v(x) + L_p(x, u(x), \dot{u}(x))\dot{v}(x) dx$$

$$\int_a^b [L_s(x, u, \dot{u})v + L_p(x, u, \dot{u})\dot{v}] dx = 0$$

First integral form of ELE. Sufficient conditions:

- L is of class C^1 ,
- u is of class C^1 .

Let us assume that v vanishes at the boundary. If we integrate by parts we obtain that

$$\int_a^b \left[L_s(x, u, \dot{u}) - \frac{d}{dx} L_p(x, u, \dot{u}) \right] v(x) dx = 0$$

Second integral form of ELE. Sufficient conditions

- L is of class C^2
- u is of class C^2

Exploiting FLCV we finally obtain

$$\frac{d}{dx} L_p(x, u, \dot{u}) = L_s(x, u, \dot{u})$$

ELE in standard differential form
(second order diff. equ.)
ODE

Oss. The LHS can be rewritten as

$$L_{px}(x, u, \dot{u}) + L_{ps}(x, u, \dot{u}) \dot{u} + L_{pp}(x, u, \dot{u}) \ddot{u} = L_s(x, u, \dot{u})$$

Neumann condition in the general setting-

$$\int_a^b L_s(x, u, \dot{u}) v + L_p(x, u, \dot{u}) \dot{v} = 0$$

$$\int_a^b L_s(x, u, \dot{u}) v - [L_p(x, u, \dot{u})]' v + [L_p(x, u, \dot{u}) v]_{x=a}^{x=b}$$

$$L_p(b, u(b), \dot{u}(b)) v(b) - L_p(a, u(a), \dot{u}(a)) v(a) = 0$$

NEUMANN CONDITION : $L_p(x, u(x), \dot{u}(x)) = 0$ in the endpoint where the condition is needed.

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Du Bois REYMOND ELE 1st integral form

$$\int_a^b L_s(x, u, \dot{u}) v + L_p(x, u, \dot{u}) \dot{v} = 0$$

↓ transform into $\dot{v}$

Let us introduce the antiderivative of $L_s(x, u, \dot{u})$, namely

$$\hat{L}(x) := \int_a^x L_s(y, u(y), \dot{u}(y)) dy \quad \forall x \in [a, b]$$

Integrating by parts we find (assume $v \equiv 0$ at boundary)

$$\int_a^b -\hat{L}(x) \dot{v} + L_p(x, u, \dot{u}) \dot{v} = 0 = \int_a^b [-\hat{L}(x) + L_p(x, u, \dot{u})] \dot{v} = 0$$

any C^1 function
with 0 average

because $\int_a^b \dot{v}(x) dx = v(b) - v(a) = 0$

DBR Lemma $\Rightarrow -\hat{L}(x) + L_p(x, u, \dot{u}) = \text{constant}$

$$L_p(x, u(x), \dot{u}(x)) = \text{const} + \int_a^x L_s(y, u(y), \dot{u}(y)) dy$$

1st order
ODE

ELE in DBR form. Does NOT require any SECOND derivative (just u and L of class C^1)

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ELE in ERDMANN FORM

We assume that the problem is AUTONOMOUS, namely L does NOT depend on x : $L(s, p)$

$$[L_p(u, \dot{u})]' \dot{u} = L_s(u, \dot{u}) \dot{u}$$

$$\text{LHS} = [L_p(u, \dot{u}) \dot{u}]' - L_p(u, \dot{u}) \ddot{u} = L_s(u, \dot{u}) \dot{u}$$

$$[L_p(u, \dot{u}) \dot{u}]' = L_s(u, \dot{u}) \dot{u} + L_p(u, \dot{u}) \ddot{u} = [L(u, \dot{u})]'$$

It follows that

$$L_p(u, \dot{u}) \dot{u} - L(u, \dot{u}) = \text{constant}$$

ERDMANN FORM

(if L is autonomous)

1st order ODE

Remark Standard and Erdmann forms are not equivalent

(1) If u satisfies ELE standard, then u satisfies ELE-ERDMANN

(2) If u satisfies ELE-ERDMANN, then u satisfies ELE stand. for all points $x \in [a, b]$ s.t. $\dot{u}(x) \neq 0$.

Generalization 1 L depending on higher order derivatives

$$F(u) = \int_a^b L(x, u, \dot{u}, \ddot{u}, \dots, u^{(k)}) dx \quad L(x, s, p_1, \dots, p_k)$$

$$\psi(t) := F(u + tv) \quad \dots \text{ same computation } \dots$$

$$\int_a^b L_s(\dots) v + \sum_{i=1}^k L_{p_i}(\dots) v^{(i)} = 0 \quad \text{First integral form}$$

$$\int_a^b L_s(\dots) v + \sum_{i=1}^k (-1)^i \frac{d^i}{dx^i} L_{p_i}(\dots) v = 0 \quad \text{Second integral form}$$

$$FLCV \Rightarrow \sum_{i=1}^k (-1)^{i+1} [L_{p_i}(\dots)]^{(i)} = L_s(\dots)$$

Generalization 2 F depends on more unknowns

$$F(u_1, \dots, u_k) = \int_a^b L(x, u_1, \dots, u_k, \dot{u}_1, \dots, \dot{u}_k) \\ L(x, s_1, \dots, s_k, p_1, \dots, p_k)$$

We use a vector of directions $(v_1, \dots, v_k)$

$$\psi(t) := F(u_1 + tv_1, \dots, u_k + tv_k)$$

$$\begin{aligned} \psi'(0) &= \int_a^b \sum_{i=1}^k [L_{s_i}(\dots) v_i + L_{p_i}(\dots) \dot{v}_i] \\ &= \int_a^b \sum_{i=1}^k [L_{s_i}(\dots) - [L_{p_i}(\dots)]'] v_i = 0 \end{aligned}$$

Use $v_1 = \text{any } C_c^\infty$ and $v_2 = \dots = v_k = 0$

$$[L_{p_i}(\dots)]' = L_{s_i}(\dots) \text{ and so on}$$

We finally obtain a system of the form

$$\left[L_{p_i}(x, u_1, \dots, u_k, \ddot{u}_1, \dots, \ddot{u}_k) \right]' = L_s(\dots) \quad i = 1, \dots, k$$

System of k ODEs of order 2.

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