

BIRTH OF BOUNDARY CONDITIONS

$$F(u) = \int_a^b (\dot{u}^2 + u^2 + \tau u) dx$$

Example 1 $X_1 = \{ u \in C^1([a,b]) : u(a) = A, u(b) = B \}$

$$V_1 = \{ v \in C^1([a,b]) : v(a) = v(b) = 0 \}$$

$$\begin{aligned} \psi(t) = F_1(u+tv) &= \int_a^b (\dot{u} + t\dot{v})^2 + (u+tv)^2 + \tau(u+tv) \\ &= \int_a^b (\dot{u}^2 + u^2 + \tau u) + t \int_a^b (2\dot{u}\dot{v} + 2uv + \tau v) + t^2 \int_a^b \dot{v}^2 + v^2 \end{aligned}$$

$$\psi'_1(0) = \int_a^b (2\dot{u}\dot{v} + 2uv + \tau v) dx$$

If u is a min point, then $\underbrace{\int_a^b (2\dot{u}\dot{v} + 2uv + \tau v) dx}_{\delta F_1(u,v)} = 0 \quad \forall v \in V_1$ Euler eqn 1st form

Integr. by parts:

$$\underbrace{\int_a^b (-2\ddot{u} + 2u + \tau) v dx}_{2^{\text{nd}} \text{ form of EULER Equ}} = 0 \quad \forall v \in V_1$$

$$\text{FLCV} \Rightarrow 2\ddot{u} = 2u + \tau$$

Euler Equ in Diff. Form

$$u(x) = c_1 e^x + c_2 e^{-x} - \frac{\tau}{2} \quad \leftarrow \text{general solution}$$

Imposing the BC $u(a) = A, u(b) = B$ we can find c_1 and c_2 in a unique way.

$$\begin{cases} 2\ddot{u} = 2u + f \\ u(a) = A \\ u(b) = B \end{cases} \quad \left. \begin{array}{l} \text{DIRICHLET} \\ \text{boundary cond.} \end{array} \right\} \quad (\text{DBC})$$

Example 2 $X_2 = \{ u \in C^1([a,b]) : u(a) = A \}$

$$V_2 = \{ v \in C^1 \dots : v(a) = 0 \}$$

As before we compute $\psi_2(x)$ and $\psi_2'(0)$, and we find

$$\int_a^b (2\ddot{u}v + 2uv + fv) = 0 \quad \forall v \in V_2 \quad \text{EU-E-1}$$

What happens when we integrate by parts?

$$0 = \int_a^b (-2\ddot{u} + 2u + f)v dx + [2\dot{u}v]_a^b \quad \forall v \in V_2$$

" $2\dot{u}(b)v(b) - \cancel{2\dot{u}(a)v(a)}$
0

Step 1 We limit ourselves to function $v \in C_c^\infty(a,b) \subseteq V_2$

We get $\int_a^b (-2\ddot{u} + 2u + f)v dx = 0 \quad \forall v \in C_c^\infty(a,b)$

FLCV $\Rightarrow 2\ddot{u} = 2u + f$ Diff. form. of Euler equ.

Step 2 Let's go back to 2nd form of EE (Euler-Lagrange)
We have that

$$0 = 2\dot{u}(b)v(b) \quad \forall v \in V_2$$

This means that $\dot{u}(b) = 0$.

We ended up with
$$\begin{cases} 2\ddot{u} = 2u + 7 \\ u(a) = A \quad \text{DBC} \\ \dot{u}(b) = 0 \quad \text{NBC (NEUMANN BC)} \end{cases}$$

Example 3 $X_3 = \{ u \in C^1([a,b]) \} = V_3$

As before, we find that

$$0 = \int_a^b (-2\ddot{u} + 2u + 7) v \, dx + \dot{u}(b) v(b) - \dot{u}(a) v(a) \quad \forall v \in V_3$$

E-E - 2nd form.

Step 1 $V_3 \subseteq C_c^\infty(a,b) \quad \text{FLCV} \Rightarrow \quad 2\ddot{u} = 2u + 7$

Step 2 Go back to 2nd form $\dot{u}(b) v(b) - \dot{u}(a) v(a) = 0$
 $\forall v \in V_3$

- Take $v \in V_3$ with $v(b) = 1$ and $v(a) = 0 \leadsto \dot{u}(b) = 0$
- Take " $v(b) = 0$ " $v(a) = 1 \leadsto \dot{u}(a) = 0$

$$\begin{cases} 2\ddot{u} = 2u + 7 \\ \dot{u}(a) = 0 \quad \text{NBC} \\ \dot{u}(b) = 0 \quad \text{NBC} \end{cases}$$

— 0 — 0 —

Example 4 $X_4 = \{ u \in C^1([a,b]) : u(a) = u(b) \} = V_4$

As before we obtain the second form of EE $\forall v \in V_4$

As in step 1 we deduce $2\ddot{u} = 2u + 7$

And in step 2 we obtain

$$\dot{u}(b) v(b) - \dot{u}(a) v(a) = 0 \quad \forall v \in V_4$$

$\overset{v(b)}{k} \quad \quad \quad \overset{v(a)}{k}$

$$k(\dot{u}(b) - \dot{u}(a)) = 0$$

Since we can take $k \neq 0$, we end up with

$$\begin{cases} \ddot{u} = 2u + 7 \\ u(a) = u(b) \\ \dot{u}(a) = \dot{u}(b) \end{cases} \quad \begin{array}{l} \text{Periodic boundary} \\ \text{conditions} \end{array} \quad (\text{PBC})$$

Example 5

$$F(u) = \int_a^b (\dot{u}^2 + 7\dot{u} + u^4) dx$$

$$X = \{u \in C^1 \dots : u(b) = B\}$$

$$\psi(t) = F(u+tv) = F(u) + t \int_a^b (2\dot{u}\dot{v} + 7\dot{v} + 4u^3v) dx + o(t)$$

$$\delta F(u, v) = \int_a^b (2\dot{u}\dot{v} + 7\dot{v} + 4u^3v) dx \quad [=0]$$

Integration by parts

$$\begin{aligned} \int_a^b (2\dot{u} + 7)\dot{v} &= [(2\dot{u} + 7)v]_a^b - \int_a^b 2\dot{u}v \\ &= \cancel{2\dot{u}(b)v(b)} + \cancel{7v(b)} - 2\dot{u}(a)v(a) - 7v(a) \end{aligned}$$

Second form of Euler equ.

$$\int_a^b (-2\ddot{u} + 4u^3)v dx - (2\dot{u}(a) + 7)v(a) = 0$$

Step 1 $v \in C_c^\infty \dots$ FLCV ... $\ddot{u} = 2u^3$

Step 2 Take $v(a) = 1$ and deduce $\dot{u}(a) = -\frac{7}{2}$

$$\begin{cases} \ddot{u} = 2u^3 \\ u(b) = B \\ \dot{u}(a) = -\frac{7}{2} \end{cases} \quad \begin{array}{l} \text{EE} \\ \text{DBC from the beginning} \\ \text{NBC born on the road} \end{array}$$

Example 6 $F(u) = \int_a^b (\ddot{u}^2 + u^3) dx$

$$X = \{u \in C^2 \dots : u(a) = A, \dot{u}(b) = B\}$$

$$\psi(t) = F(u) + t \int_a^b (2\ddot{u}\ddot{v} + 3u^2v) dx + o(t)$$

$$\delta F(u, v) = \int_a^b (2\ddot{u}\ddot{v} + 3u^2v) dx$$

$$\begin{aligned} \int \ddot{u}\ddot{v} &= [\ddot{u}\dot{v}]_a^b - \int_a^b u''' \dot{v} \\ &= [\ddot{u}\dot{v}]_a^b - [u'''v]_a^b + \int_a^b u^{iv} v \end{aligned}$$

Second form of Euler equ

$$\int_a^b [2u^{iv} + 3u^2]v dx + 2[\ddot{u}\dot{v}]_a^b - 2[u'''v]_a^b$$

Step 1 $\rightsquigarrow 2u^{iv} = -3u^2$
 $u(a) = A$
 $\dot{u}(b) = B$

We need two more condition

Step 2 $\rightsquigarrow \cancel{\ddot{u}(b)\dot{v}(b)} - \ddot{u}(a)\dot{v}(a) - \cancel{u'''(b)v(b)} + \cancel{u'''(a)v(a)} = 0$
 $\overset{0}{\underset{0}{}}$ $\overset{0}{\underset{0}{}}$

Take $v \in V$ with $\dot{v}(a) = 0$ and $v(b) = 1 \rightsquigarrow u'''(b) = 0$
 $\dot{v}(a) = 1$ and $v(b) = 0 \rightsquigarrow \ddot{u}(a) = 0$

$\ddot{u}(a) = 0$
 $u'''(b) = 0$ } Two extra conditions born on the road.