

FUNDAMENTAL LEMMA IN THE CALC. VAR.

Main idea: suppose $f: [a, b] \rightarrow \mathbb{R}$ (continuous)

Assume that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in \text{some class of fcts.}$$

Then, under suitable assumptions, $f(x) = 0$ for every $x \in [a, b]$.

Def. We say that $v \in C_c^\infty(a, b)$ if

$\uparrow$
compact support

$v \in C^\infty$ in (a, b) and there exists $[c, d] \subseteq (a, b)$ such that

$$v(x) = 0 \quad \forall x \notin [c, d]$$

($v \equiv 0$ in a neighborhood of $\partial[a, b]$)

LEMMA (FLCV) Suppose $f: [a, b] \rightarrow \mathbb{R}$ continuous such that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in C_c^\infty(a, b)$$

Then $f(x) = 0$ for every $x \in [a, b]$.

Two strategies

① by contradiction

② by density of C_c^∞ in C^0 .

Proof 1] Assume that $f \not\equiv 0$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) \neq 0$.

WLOG I can assume $x_0 \in (a, b)$ and $f(x_0) > 0$.

Then there exist $\delta > 0$ and $r > 0$ such that

$$[x_0 - r, x_0 + r] \subseteq (a, b)$$

$$f(x) \geq \delta \quad \forall x \in [x_0 - r, x_0 + r]$$

Standard exercise: there exists

$\psi \in C_c^\infty(a, b)$ such that

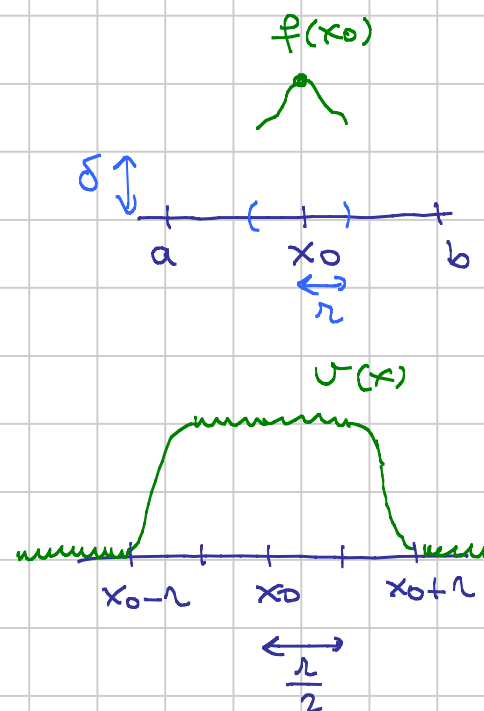
$$\bullet \psi(x) = 1 \quad \text{on } [x_0 - \frac{r}{2}, x_0 + \frac{r}{2}]$$

$$\bullet \psi(x) = 0 \quad \text{outside } [x_0 - r, x_0 + r]$$

$$\bullet 0 \leq \psi(x) \leq 1 \quad \forall x \in \mathbb{R}$$

Now

$$\begin{aligned} \int_a^b f(x) \psi(x) dx &= \int_{x_0-r}^{x_0+r} f(x) \psi(x) dx \\ &\geq \int_{x_0-\frac{r}{2}}^{x_0+\frac{r}{2}} f(x) \psi(x) dx \\ &= \int_{x_0-\frac{r}{2}}^{x_0+\frac{r}{2}} \underbrace{f(x)}_{\geq \delta} dx \geq r\delta > 0 \end{aligned}$$



Proof 2 General fact: for every continuous function

$g: [a, b] \rightarrow \mathbb{R}$ there exists a sequence of functions $g_n \in C_c^\infty(a, b)$ such that

(i) $\exists M \in \mathbb{R}$ s.t. $|g_n(x)| \leq M$ for every $x \in [a, b]$, $n \in \mathbb{N}$

(ii) $g_n \rightarrow g$ uniformly on compact subsets of (a, b)

Use this fact with $g = f$ and use $\psi(x) \equiv g_n(x)$

$C_c^\infty(a, b)$

$g_n \rightarrow f$

$$0 = \int_a^b f(x) g_n(x) dx \longrightarrow \int_a^b f(x)^2 dx$$

↑
because of
(i) and (ii)

Since $f(x)$ is continuous, this is possible only if $f \equiv 0$.

Du Bois - REYMOND LEMMA

Assume $f: [a, b] \rightarrow \mathbb{R}$ cont.

Assume

$$(*) \quad \int_a^b f(x) v(x) dx = 0 \quad \forall v \in C_c^\infty(a, b) \text{ s.t. } \int_a^b v(x) dx = 0$$

Then $f(x)$ is constant in $[a, b]$.

Brutal proof Assume we can integrate by parts

$$0 = \int f v = - \int \dot{f} V \quad \text{where } V'(x) = v(x) \text{ in } (a, b) \text{ and } V(a) = V(b) = 0$$

Then FLCV $\Rightarrow \dot{f} \equiv 0 \Rightarrow f$ is constant.

True proof. If f satisfies $(*)$, then $f + c$ satisfies $(*)$ for every constant c .

Let us assume that f is not constant. Then there exist x_0 and y_0 in (a, b) such that $f(x_0) \neq f(y_0)$.

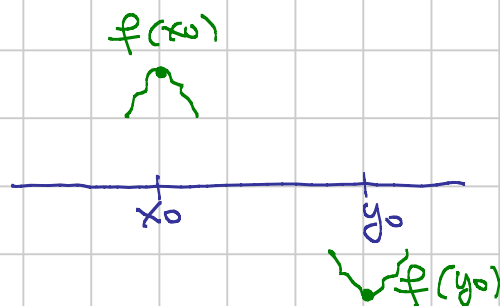
Up to choosing the right constant c , I can assume that

$$f(x_0) = -f(y_0)$$

Choose $r > 0$ and $\delta > 0$ s.t.

$$f(x) \geq \delta \text{ in } [x_0 - r, x_0 + r]$$

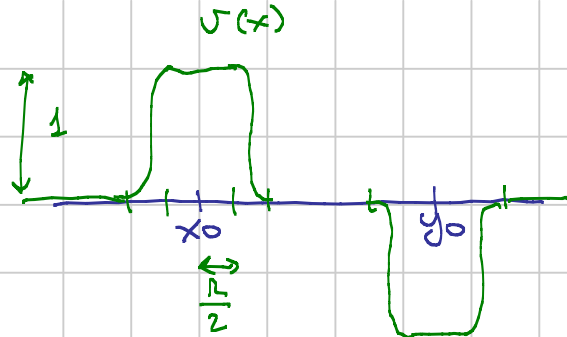
$$f(x) \leq -\delta \text{ in } [y_0 - r, y_0 + r]$$



Finally, choose v this way $\leadsto$

Now $\int_a^b v(x) dx = 0$, but

$$\int_a^b f(x) v(x) dx > 0.$$



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Remark I can also provide a second proof as in the case of FLCV
Now I need an approx. with extra requirement that

$$\int g(x) dx = \int g_n(x) dx = 0.$$

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Remark Let $f: [a, b] \rightarrow \mathbb{R}$ be given. Let us consider

$$V = \left\{ v: [a, b] \rightarrow \mathbb{R} : v \text{ is cont. and } \int_a^b f(x) v(x) dx = 0 \right\}$$

Then V has the following properties

(1) V is a vector space

(2) V is closed under uniform convergence, namely if $v_m \rightarrow v_\infty$ unif. and $v_m \in V \Rightarrow v_\infty \in V$

(3) V is closed under unif. conv. in compact subsets of $[a, b] \setminus S$ (where S is a finite set) provided that $v_m \rightarrow v_\infty$ with v_m equibounded.

Consequence In order to conclude that $f \equiv 0$ in $[a, b]$ it is enough to check that

$$\int_a^b f(x) v(x) dx = 0 \quad \forall v \in F$$

better: the closure of $\text{span} F$

where F is a set of functions whose closure[↑] with respect to (3) is dense in $C^0([a, b])$ (if $\text{span} F$ is dense in C_c^∞ , then it is dense in C^0).

Example 1 $\min \left\{ \underbrace{\int_a^b \dot{u}^2 dx}_{F(u)} : \begin{array}{l} u \in C^1([a,b]) \\ u(a) = A \quad u(b) = B \\ \int_a^b u(x) dx = 8 \end{array} \right\}$

X

The space X is affine with reference vector space

$$V = \left\{ v \in C^1 : v(a) = v(b) = 0, \int_a^b v(x) dx = 0 \right\}$$

$$\psi(t) := F(u + tv) = F(u) + 2t \int_a^b \dot{u} \dot{v} dx + t^2 \int_a^b \dot{v}^2 dx$$

$$\delta F(u, v) = \psi'(0) = 2 \underbrace{\int_a^b \dot{u} \dot{v} dx}_{\substack{\text{Euler equ} \\ 1^{st} \text{ form.}}} \quad \forall v \in V$$

Integrating by parts we obtain

$$\int_a^b \ddot{u} v dx = 0 \quad \forall v \in V$$

$$\text{DBR Lemma} \Rightarrow \ddot{u} = \text{constant} \Rightarrow u(x) = ax^2 + bx + c$$

Easy to see that $BC + \int u = 8$ allow to determine a, b, c in a unique way.

In the usual way one can prove that u is actually a minimum point.

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