

INDIRECT METHOD

General setting X set $f: X \rightarrow \mathbb{R}$ function
 $\min \{f(x) : x \in X\}$

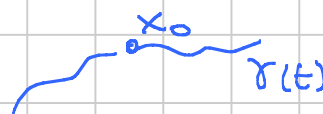
Let $x_0 \in X$ be any point. Let $\delta > 0$, and let $\gamma: (-\delta, \delta) \rightarrow X$ be curve such that $\gamma(0) = x_0$.

I can consider

$$\psi: (-\delta, \delta) \rightarrow \mathbb{R}$$

defined as

$$\psi(t) = f(\gamma(t)) \quad \forall t \in (-\delta, \delta)$$



Trivial fact Assume that $x_0 \in \arg\min \{f(x) : x \in X\}$.

Then $t=0$ is a minimum point for $\psi(t)$.

In particular

$$\psi'(0) = 0 \quad \text{if it exists}$$

Def. We define the FIRST VARIATION of f in the point x_0 w.r.t the curve γ as $\psi'(0)$.

$$\delta f(x_0, \gamma) := \psi'(0) \quad (\text{provided it exists})$$

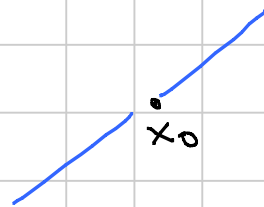
Theorem (Necessary condition) If x_0 is a min. point, then

$$\delta f(x_0, \gamma) = 0 \quad \text{for every curve } \gamma: (-\delta, \delta) \rightarrow X \\ \text{with } \gamma(0) = x_0$$

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Special case 1 $X = \text{vector space } V$

Take $\gamma(t) := x_0 + t v$
 $\uparrow$
 $v \in V$



$$\begin{aligned} \delta f(x_0, v) &= \text{derivative of } f(x_0 + t v) \text{ computed at } t=0 \\ &\quad \uparrow \\ &\quad \text{direction} \\ &= \lim_{t \rightarrow 0} \frac{f(x_0 + t v) - f(x_0)}{t} \\ &= \text{Derivative of } f(x) \text{ in the direction } v \end{aligned}$$

Special case 2 $X = \text{affine space with reference } \overbrace{\text{space}}^{\text{vector}} V$

Once again if $x_0 \in X$ and $v \in V$ we have that $x_0 + t v \in X$ and so we can define

$$\delta f(x_0, v)$$

$\uparrow \quad \uparrow$
 $\in X \quad \in V$

as before

Examples $X = C^1([a, b]) = \text{vector space}$

$$X = \{ u \in C^1([a, b]) : u(a) = A \}$$

is an affine space with reference space

$$V = \{ u \in C^1([a, b]) : u(a) = 0 \}$$

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Three main examples

$$F_1(u) = \int_a^b \dot{u}^2 dx, \quad F_2(u) = \int_a^b |\dot{u}| dx, \quad F_3(u) = \int_a^b |\dot{u}|^{1/2} dx$$

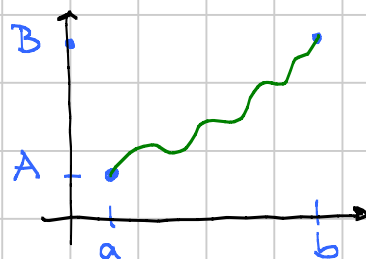
$$\min \begin{cases} F_1(u) \\ F_2(u) \\ F_3(u) \end{cases} : u \in C^1([a,b]), u(a)=A, u(b)=B$$

$\uparrow$ given numbers $\uparrow$

Let us consider $F_1(u)$.

X is an affine space based on

$$V = \{ u \in C^1 \dots : u(a) = u(b) = 0 \}$$



Let u be a minimum point. Let $v \in V$ be any direction

$$\begin{aligned} \psi(t) &:= F(u+tv) = \int_a^b (\dot{u} + t\dot{v})^2 dx \\ &= \int_a^b \dot{u}^2 + 2t\dot{u}\dot{v} + t^2\dot{v}^2 dx \\ &= \int_a^b \dot{u}^2 + 2t \int_a^b \dot{u}\dot{v} + t^2 \int_a^b \dot{v}^2 \end{aligned}$$

$$\delta F_1(u, v) = \psi'(0) = 2 \int_a^b \dot{u}\dot{v}$$

If u is a minimum point, then

$$\int_a^b \dot{u}\dot{v} = 0 \quad \forall v \in V$$

EULER Equ, 1st form

Integrating by parts we obtain that

$$\begin{aligned} \int_a^b \dot{u}\dot{v} dx &= [\dot{u}(x)v(x)]_a^b - \int_a^b \ddot{u}(x)v(x) dx \\ &= \underbrace{\dot{u}(b)}_0 v(b) - \underbrace{\dot{u}(a)}_0 v(a) - \int_a^b \ddot{u} v dx \end{aligned}$$

$\uparrow$ derivative $\uparrow$ integrate

If u is a min point AND $u \in C^2([0,1])$
then necessarily

$$\int_a^b \ddot{u} v \, dx = 0 \quad \forall v \in V$$

EULER EQN, 2nd integral form

Seems reasonable to guess that this is possible only if $\ddot{u} = 0$

$$\ddot{u}(x) = 0 \quad \forall x \in [a,b]$$

EULER EQN
in differential form

The solutions of our pbm is probably the unique function of the form

$$u(x) = mx + n$$

satisfying the BC (boundary conditions).

Formal proof that $u_0(x) = mx + n$, with suitable values of m and n , is the unique min. point for F_1 in X .

Take any $u \in X$. We can write $u = u_0 + v$ for a suitable $v \in V$. Then as before

$$\begin{aligned} F_1(u) &= \int_a^b (\ddot{u}_0 + \ddot{v})^2 \, dx = \int_a^b \ddot{u}_0^2 \, dx + 2 \int_a^b \ddot{u}_0 \ddot{v} \, dx + \int_a^b \ddot{v}^2 \, dx \\ &\quad \underbrace{\int_a^b \ddot{u}_0^2 \, dx}_{F_1(u_0)} \quad \underbrace{2 \int_a^b \ddot{u}_0 \ddot{v} \, dx}_{\begin{array}{l} \parallel \leftarrow u_0 \in C^2 \text{ and } v \equiv 0 \\ \text{at } a \text{ and } b \\ = -2 \int_a^b \ddot{u}_0 v \, dx \\ \parallel \\ 0 \end{array}} + \int_a^b \ddot{v}^2 \, dx \\ &= F_1(u_0) + \int_a^b \ddot{v}^2 \, dx \end{aligned}$$

$$\geq F_1(u_0)$$

with equality $\Leftrightarrow \int_a^b \ddot{v}^2 \, dx = 0 \Leftrightarrow \ddot{v}(x) \equiv 0 \Leftrightarrow v(x) \equiv 0$
 $\uparrow$
 $v(a) = v(b) = 0$

In the case of F_2 , u_0 is a minimum point again, but it is not unique.

Take any $u \in X$. Let us assume wlog that $B \geq A$

$$F_2(u) = \int_a^b |\dot{u}(x)| dx \geq \int_a^b \dot{u}(x) dx = u(b) - u(a) = B - A \stackrel{''}{=} F(u_0)$$

BUT any nondecreasing function u is again a minimum point for F_2 because

$$F_2(u) = \int_a^b |\dot{u}(x)| dx = \int_a^b \dot{u}(x) dx = B - A = F(u_0)$$

↑
because $\dot{u}(x) \geq 0$ for every $x \in [a, b]$

This is $\Leftrightarrow$ because if $\dot{u}(x) < 0$ somewhere, then we have strict inequality.
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In the case of $F_3(u)$, the minimum pbm. has no solution unless $A = B$.

I claim that

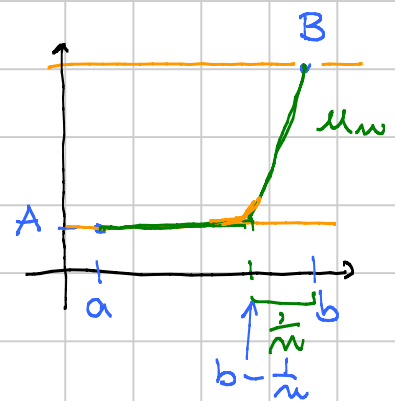
$$\inf \{ F_3(u) \mid u \in C^1([a, b]) + B.C. \} = 0$$

This proves the the min does NOT exist

$$\begin{aligned} F_3(u_n) &= \int_a^b |\dot{u}_n(x)|^{1/2} dx \\ &= \int_{b-\frac{1}{n}}^b |\dot{u}_n(x)|^{1/2} dx \end{aligned}$$

$$= \int_{b-\frac{1}{n}}^b |n(B-A)|^{1/2} dx$$

$$= |B-A|^{1/2} \sqrt{n} \frac{1}{n} = \frac{|B-A|^{1/2}}{\sqrt{n}} \rightarrow 0$$



What is important is the convexity of L with resp. to p