

Program

- ① Indirect method
- ② Direct method
- ③ Relaxation
- ④ Gamma - convergence

First part: give an idea of ① - ④
 Second part: more details and final examples

so final examples.

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What is calculus of variations? Minimum problems

$$\min \{ f(x) : x \in X \} \quad f: X \rightarrow \mathbb{R}$$

$\uparrow$ minimum value $\uparrow$ any set

$$\begin{aligned} \text{Minimum points} &= \{ x \in X : f(x) \text{ attains the minimum} \} \\ &= \operatorname{argmin} \{ f(x) : x \in X \} \subseteq X \end{aligned}$$

"Discussing" a minimum problem means

- ① Proving or disproving that min exists
- ② If it exists, finding the minimum value and argmin
- ③ If it does not, understand why.

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Trivial example $\min \{ \underbrace{x^2 + 5x}_{f(x)} : x \in \mathbb{R} \}$

Indirect method: $f'(x) = 2x + 5$, $f'(x) = 0 \Leftrightarrow x = -\frac{5}{2}$
 guess: the min. is achieved when $x = -\frac{5}{2}$ and
 $f(-\frac{5}{2}) = \text{min. value} = \frac{25}{4} - \frac{25}{2} = -\frac{25}{4}$

Rigorous proof: want to prove that $f(x) \geq -\frac{25}{4} \quad \forall x \in \mathbb{R}$

$$x^2 + 5x + \frac{25}{4} \geq 0 \quad \forall x \in \mathbb{R} \quad \left(x + \frac{5}{2}\right)^2 \geq 0$$

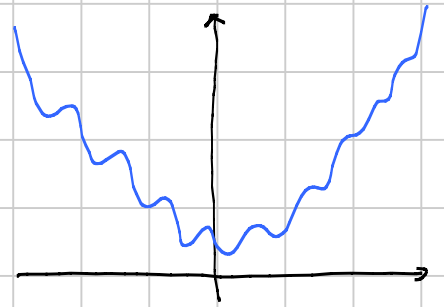
The last ineq. is true for every $x \in \mathbb{R}$, with equality $\Leftrightarrow x = -\frac{5}{2}$

Direct method: proving existence by means of general results

Consider $f(x)$. Observe that $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = +\infty$

and that f is continuous in $\mathbb{R}$.

This guarantees the existence of the minimum.



Relaxation $\min \underbrace{\{(x^2-3)^2\}}_{f(x)} : x \in \mathbb{Q}$

$f(x) \geq 0 \quad \forall x \in \mathbb{Q}$, but $f(x) = 0$ has no solution in $\mathbb{Q}$ and actually

$$\inf \{(x^2-3)^2 : x \in \mathbb{Q}\} = 0$$

and the "minimizing sequences" are those converging to $\pm\sqrt{3}$.

When min does not exist, then the goal is

→ finding $\inf$

→ characterizing sequences $\{x_n\}$ with $f(x_n) \rightarrow \inf$.

Gamma convergence. Sequence of min pbms depending on a parameter $n \in \mathbb{N}$, for example

$$M_n = \min \underbrace{\{x^2 - \sin x + x^{2n}\}}_{f_n(x)} : x \in \mathbb{R}$$

Questions:

1 - $M_n \rightarrow ?$ as $n \rightarrow +\infty$

2 - what is the limit of minimum points?

Idea: $M_n \rightarrow \min \{x^2 - \sin x : x \in [-1, 1]\}$

We will say that

$$f_n(x) \xrightarrow{\text{Gamma}} f_\infty(x) = \begin{cases} x^2 - \sin x & \text{if } x \in [-1, 1] \\ +\infty & \text{otherwise} \end{cases}$$

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INTEGRAL FUNCTIONALS

X = space of functions, for example

$$X = \{u \in C^3([0, 2]) : u(0) = 7\}$$

$F: X \rightarrow \mathbb{R}$ is usually called functional
(function defined in a space of functions)

For example

$$F(u) = \int_0^2 \{[u'(x)]^2 + 3 \cos(u(x))\} dx$$

↑
integral functional

We consider integral functionals of the following form

$$F(u) = \int_a^b L(x, u(x), u'(x)) dx$$

$$L(x, s, p)$$

where $u: [a, b] \rightarrow \mathbb{R}$

and

$$L: \underbrace{[a, b]}_x \times \underbrace{\mathbb{R}}_s \times \underbrace{\mathbb{R}}_p$$

More general examples

$$F(u) = \int_a^b L(x, u', u'', \dots, u^{(n)}) dx$$

$$F(u, v) = \int_a^b L(x, u, v, u', v') dx$$

$$F(u) = \int_a^b L(x, u, u') dx \quad u: [a, b] \rightarrow \mathbb{R}^2$$

$$F(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) dx \quad u: \Omega \rightarrow \mathbb{R} \text{ but } \Omega \subseteq \mathbb{R}^m$$

This course: $u: [a, b] \rightarrow \mathbb{R}$ and sometimes $u: [a, b] \rightarrow \mathbb{R}^k$

In one lecture $u: \Omega \rightarrow \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^m$

More difficult: case where $u: \Omega \rightarrow \mathbb{R}^k$, with $n > 1, k > 1$

Which classes of functions $u: [a, b] \rightarrow \mathbb{R}$ do we consider?

Example $F(u) = \int_0^1 [\dot{u}^2 - xu] dx$

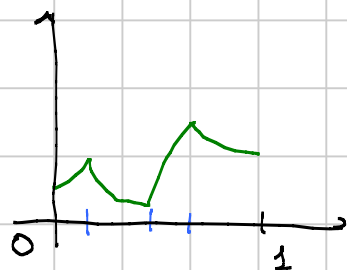
$$\min \{ F(u) : u \in C^1([0, 1]) \}$$

$$\min \{ F(u) : u \in C^{1/4}([0, 1]) \} \quad \text{we ask more regularity}$$

We can ask less regularity, for example $u \in \text{PWC}^1([0, 1])$

Def. $u \in \text{PWC}^1([0, 1])$ if there exists $0 = x_0 < x_1 < \dots < x_n = 1$ such that

- $u \in C^1([x_{k-1}, x_k])$ for every $k = 1, \dots, n$
- $u \in C^0([0, 1])$



We can interpret $F(u)$ as

$$F(u) = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} [u^2 - xu] dx$$

Quite general fact Let us consider $F(u) = \int_a^b L(x, u, u') dx$

with $L: [a, b] \times \mathbb{R} \times \mathbb{R}$ continuous. Then

$$\inf \{F(u) : u \in C^k([a, b])\} \stackrel{\geq}{=} \inf \{F(u) : u \in C^1([a, b])\} \\ \stackrel{=}{=} \inf \{F(u) : u \in \text{PWC}^1([a, b])\}$$

[Proof : we can approximate every $u \in \text{PWC}^1([a, b])$ with a sequence $u_n \rightarrow u$ in such a way that

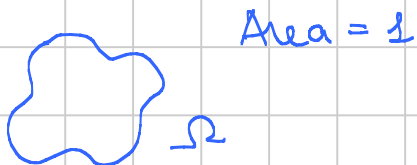
$$F(u_n) \rightarrow F(u)$$

We can also take $u_n \in C^\infty([a, b])$.]
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Problems we can NOT consider $\leadsto$ many...

in particular geometric pbms, namely problems in which the unknowns is a set

Example $\min \{ \text{length}(\partial\Omega) : \Omega \subseteq \mathbb{R}^2 \text{ is a set with } \text{meas}(\Omega) = 1 \}$



$F = \boxed{\text{length}}$ of boundary
 $\uparrow$ to be defined

$X =$ set of subsets of $\mathbb{R}^2$ with given Area
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