

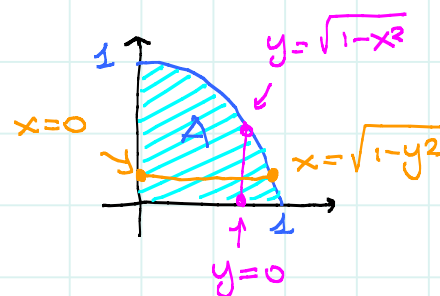
INTEGRALI DOPPI IN COORDINATE POLARI

$$\iint_A f(x,y) dx dy = \iint_B f(\rho \cos \theta, \rho \sin \theta) \rho d\rho d\theta$$

insieme A
descritto in coord. polari

raggiamento

Esempio 1 $\iint_A y dx dy$



Osservo che A si scrive bene in
coord. polari

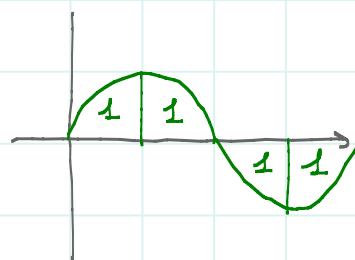
$$A = \{(\rho \cos \theta, \rho \sin \theta) : \rho \in [0, 1], \theta \in [0, \frac{\pi}{2}]\} \leftarrow \text{nelle variabili } \rho \text{ e } \theta \text{ è un rettangolo}$$

$$\iint_A y dx dy = \int_0^1 d\rho \int_0^{\frac{\pi}{2}} d\theta \quad \underbrace{\rho \sin \theta}_{y \text{ in coord. pol.}} \quad \underbrace{\rho}_{\text{ragg.}}$$

A descritto in coord. polari

$$= \int_0^1 \rho^2 d\rho \underbrace{\int_0^{\frac{\pi}{2}} \sin \theta d\theta}_1 = \int_0^1 \rho^2 d\rho = \frac{1}{3}$$

Analisi 1 : le "semicodine" di sin e cos
hanno area 1



2° modo A è normale risp. asse x

$$A = \{(x,y) \in \mathbb{R}^2 : x \in [0, 1], 0 \leq y \leq \sqrt{1-x^2}\}$$

$$\begin{aligned}
 \iint_A y \, dx \, dy &= \int_0^1 dx \int_0^{\sqrt{1-x^2}} y \, dy \\
 &= \int_0^1 dx \left[\frac{1}{2} y^2 \right]_{y=0}^{y=\sqrt{1-x^2}} = \frac{1}{2} \int_0^1 (1-x^2) \, dx \\
 &= \frac{1}{2} \left[x - \frac{1}{3} x^3 \right]_{x=0}^{x=1} = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} \quad \checkmark
 \end{aligned}$$

3° modo A è normale risp. asse y

$$A = \{ (x, y) \in \mathbb{R}^2 : y \in [0, 1], 0 \leq x \leq \sqrt{1-y^2} \}$$

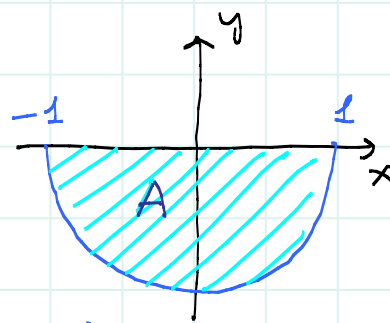
$$\iint_A y \, dx \, dy = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx \, y = \int_0^1 y \, dy \int_0^{\sqrt{1-y^2}} 1 \, dx$$

$$= \int_0^1 y \sqrt{1-y^2} \, dy = \left[-\frac{1}{3} (1-y^2)^{\frac{3}{2}} \right]_{y=0}^{y=1} = \frac{1}{3} \quad \checkmark$$

Verifica della primitiva: $\left[-\frac{1}{3} (1-y^2)^{\frac{3}{2}} \right]' = -\frac{1}{3} \cdot \frac{3}{2} (1-y^2)^{\frac{1}{2}} (-2y) = y \sqrt{1-y^2} \quad \checkmark$

— o — o —

Esempio 2 $\iint_A x^2 \, dx \, dy$



Osservo che A in coord. polari si descrive come
 $\rho \in [0, 1], \theta \in [\pi, 2\pi]$ (volendo $\theta \in [-\pi, 0]$)

1° modo normale asse x

$$A = \{ (x, y) \in \mathbb{R}^2 : x \in [-1, 1], -\sqrt{1-x^2} \leq y \leq 0 \}$$

$$\iint_A x^2 \, dx \, dy = \int_{-1}^1 x^2 \, dx \int_{-\sqrt{1-x^2}}^0 1 \, dy = \int_{-1}^1 x^2 \sqrt{1-x^2} \, dx = \text{uhm...}$$

Analisi 1

$$\int_a^b 1 dx = (b-a) = \text{lungh. intervallo}$$

2° modo coord. polari

$$\iint_A x^2 dx dy = \underbrace{\int_0^1 dp \int_{\pi}^{2\pi} d\theta}_{\substack{A \text{ in coord.} \\ \text{polari}}} \underbrace{p^2 \cos^2 \theta}_{x^2} \underbrace{p}_{\text{rag}}$$

$$= \int_0^1 p^3 dp \underbrace{\int_{\pi}^{2\pi} \cos^2 \theta d\theta}_{\frac{\pi}{2}} = \frac{\pi}{2} \int_0^1 p^3 dp = \frac{\pi}{8}$$

Analisi 1 Se a e b sono multipli interi di $\frac{\pi}{2}$, allora

$$\int_a^b \cos^2 \theta d\theta = \int_a^b \sin^2 \theta d\theta = \frac{1}{2} (b-a) = \text{metà lungh. intervallo}$$

Infatti $\int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$ perché è la stessa figura simmetrizzata

La somma dei due è l'integrale di $\sin^2 \theta + \cos^2 \theta = 1$, cioè la lungh. dell'intervallo.

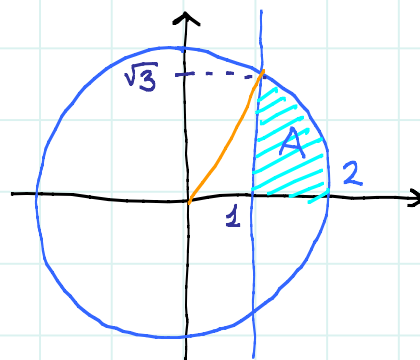
Poi i grafici si ripetono per simmetria.

Torniamo a

$$\begin{aligned} \int_{-1}^1 x^2 \sqrt{1-x^2} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cdot \cos t \cdot \cos t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cdot \cos^2 t dt \\ &\quad \begin{matrix} \uparrow \\ x = \sin t \\ dx = \cos t dt \end{matrix} \\ &= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(2t) dt = \frac{1}{8} \int_{-\pi}^{\pi} \sin^2 z dz \\ &\quad \begin{matrix} \uparrow \\ z = 2t \\ dz = 2dt \end{matrix} \\ &= \frac{1}{8} \int_{-\pi}^{\pi} \sin^2 z dz \end{aligned}$$

Esempio 3 $A = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, x \geq 1, y \geq 0 \}$

$$\iint_A x \, dx \, dy$$



1° modo Normale asse x

$$\begin{aligned} \iint_A x \, dx \, dy &= \int_1^2 dx \int_0^{\sqrt{4-x^2}} dy \quad x \\ &= \int_1^2 x \sqrt{4-x^2} \, dx = \text{si fa} \end{aligned}$$

2° modo Normale asse y

$$\iint_A x \, dx \, dy = \int_0^{\sqrt{3}} dy \int_0^{\sqrt{4-y^2}} x \, dx = \frac{1}{2} \int_0^{\sqrt{3}} (4-y^2) \, dy = \text{si fa}$$

3° modo scomporre

$$\iint_A x \, dx \, dy = \iint_S x \, dx \, dy - \iint_T x \, dx \, dy \quad \text{dove}$$

$$S = \{ (x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, \frac{\pi}{3}] \}$$

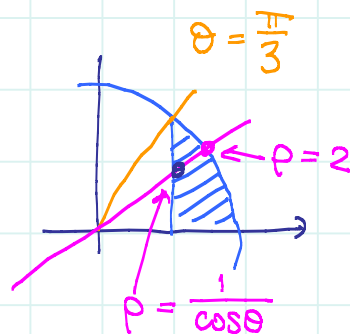
$$T = \{ (x, y) \in \mathbb{R}^2 : x \in [0, 1], 0 \leq y \leq \sqrt{3}x \}$$

Quindi entrambi si fanno molto comodamente

4° modo Polari dirette su A

Bisogna descrivere A in coord. polari

$$\begin{array}{lll} x^2 + y^2 \leq 4 & y \geq 0 & x \geq 1 \\ \downarrow & \downarrow & \downarrow \\ \rho \in [0, 2] & \theta \in [0, \pi] & \rho \cos \theta \geq 1 \end{array} \quad \begin{array}{l} \text{[versi} \\ \text{connetti dopo} \\ \text{video]} \end{array}$$



$$\iint_A x \, dx \, dy = \int_0^{\frac{\pi}{3}} d\theta \int_{\frac{1}{\cos \theta}}^2 \rho \cos \theta \cdot \rho \, d\rho = \text{si fa.}$$