

INTEGRALI DOPPI IN COORDINATE POLARI

$$\iint_A f(x,y) dx dy = \iint_B f(\rho \cos \theta, \rho \sin \theta) \underbrace{\rho}_{\text{pagamento}} d\rho d\theta$$

$\uparrow$
 insieme A descritto
 in coord. polari

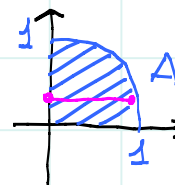
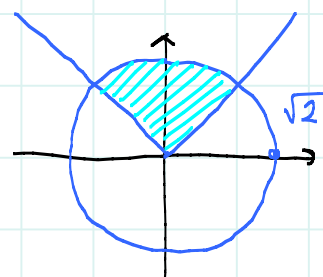
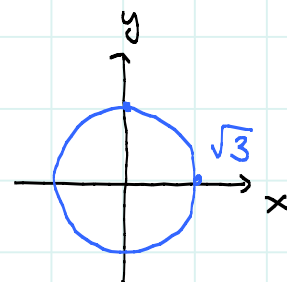
Descrizione di insiemi in coord. polari

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 3\}$$

$$B = \{(\rho, \theta) : 0 \leq \rho \leq \sqrt{3}, 0 \leq \theta \leq 2\pi\}$$

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2, y \geq |x|\}$$

$$B = \{(\rho, \theta) \in \mathbb{R}^2 : 0 \leq \rho \leq \sqrt{2}, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}\}$$

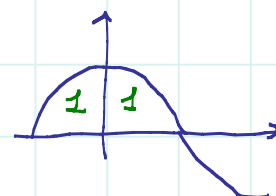


Esempio 1 $\iint_A x dx dy$

$$B = \{(\rho, \theta) \in \mathbb{R}^2 : 0 \leq \rho \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\} \leftarrow B \text{ è un rettangolo}$$

1° modo: in coord. polari

$$\iint_B \underbrace{\rho \cos \theta}_x \underbrace{\rho}_{\text{pag.}} d\rho d\theta = \underbrace{\int_0^1 \rho^2 d\rho}_{\frac{1}{3}} \underbrace{\int_0^{\frac{\pi}{2}} \cos \theta d\theta}_1 = \frac{1}{3}$$



Fatto di Analisi 1: le "colline" di $\sin \theta$ e $\cos \theta$ hanno area 2

2° modo: normale asse x

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$$

$$\begin{aligned}\iint_A x \, dx \, dy &= \int_0^1 dx \int_0^{\sqrt{1-x^2}} x \, dy = \int_0^1 x \int_0^1 1 \, dy \\ &= \int_0^1 x \sqrt{1-x^2} \, dx = \left[-\frac{1}{3} (1-x^2)^{\frac{3}{2}} \right]_{x=0}^{x=1} = \frac{1}{3} \quad \text{☺}\end{aligned}$$

verificare la primitiva derivando!

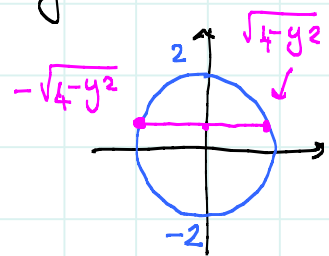
3° modo : normale risp. asse y

$$A = \{ (x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, 0 \leq x \leq \sqrt{1-y^2} \}$$

$$\begin{aligned}\iint_A x \, dx \, dy &= \int_0^1 dy \int_0^{\sqrt{1-y^2}} x \, dx = \int_0^1 \frac{1}{2} [x^2]_{x=0}^{x=\sqrt{1-y^2}} dy \\ &= \frac{1}{2} \int_0^1 (1-y^2) dy = \frac{1}{2} \left[y - \frac{1}{3} y^3 \right]_{y=0}^{y=1} = \frac{1}{2} \left[1 - \frac{1}{3} \right] = \frac{1}{3} \quad \text{☺}\end{aligned}$$

Esempio 2 $A = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \}$ $\iint_A y^2 \, dx \, dy$

1° modo : coord. polari



$$\begin{aligned}\underbrace{\int_0^2 dp \int_0^{2\pi} d\theta}_{A \text{ in coord. polari}} \underbrace{p^2 \sin^2 \theta}_{y^2} \underbrace{p}_{\text{pag.}} &= \int_0^2 p^3 dp \int_0^{2\pi} \sin^2 \theta \, d\theta \\ &= \pi \int_0^2 p^3 dp = \pi \left[\frac{1}{4} p^4 \right]_{p=0}^{p=2} = 4\pi.\end{aligned}$$

Fatto di analisi 1

$$\int_a^b \sin^2 \theta \, d\theta = \int_a^b \cos^2 \theta \, d\theta = \frac{b-a}{2}$$

se a e b sono multipli interi di $\frac{\pi}{2}$

Basta vederlo nel caso in cui $a=0$ e $b=\frac{\pi}{2}$ (per simmetria)

$$S = \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \quad C = \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \quad \rightsquigarrow \quad S = C \text{ per simmetria}$$

$$S + C = \int_0^{\frac{\pi}{2}} 1 \, d\theta = \frac{\pi}{2}$$

2° modo: normale axe y

$$A = \{(x,y) \in \mathbb{R}^2 : -2 \leq y \leq 2 ; -\sqrt{4-y^2} \leq x \leq \sqrt{4-y^2}\}$$

$$\iint_A y^2 dx dy = \int_{-2}^2 dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y^2 dx$$

$$= \int_{-2}^2 y^2 dy \cdot 2\sqrt{4-y^2} = 2 \int_{-2}^2 y^2 \sqrt{4-y^2} dy$$

$$\begin{aligned} y &= 2 \sin t \quad \nearrow \\ dy &= 2 \cos t dt \end{aligned} \quad \Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cdot 4 \sin^2 t \sqrt{4-4\sin^2 t} dt$$

$$= 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cdot \cos t \cdot 2 \cos t dt$$

$$= 32 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt = 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(2t) dt$$

$$\begin{aligned} 2t &= w \quad \nearrow \\ 2dt &= dw \end{aligned} \quad \Rightarrow \quad 8 \int_{-\pi}^{\pi} \sin^2 w \cdot \frac{1}{2} dw = 4 \underbrace{\int_{-\pi}^{\pi} \sin^2 w dw}_{\pi} = 4\pi \quad \text{😊}$$

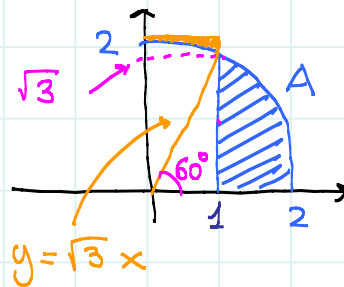
Esempio 3 $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, x \geq 1, y \geq 0\}$

$$\iint_A x dx dy$$

1° modo Normale axe x

$$A = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 2, 0 \leq y \leq \sqrt{4-x^2}\}$$

$$\int_1^2 dx \int_0^{\sqrt{4-x^2}} dy \cdot x = \text{si fa}$$



2° modo Normale asse y

$$A = \{ (x, y) \in \mathbb{R}^2 : 0 \leq y \leq \sqrt{3}, 1 \leq x \leq \sqrt{4-y^2} \}$$

3° modo $\iint_A \dots = \iint_{\text{setto}} \dots - \iint_{\text{triangolo}} \dots$

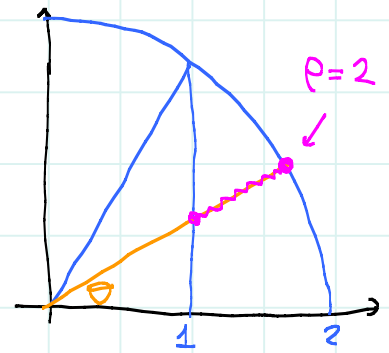
$$\iint_{\text{setto}} x \, dx \, dy = \int_0^2 dp \int_0^{\frac{\pi}{3}} p \cos \theta \cdot p \cdot d\theta = \text{banale}$$

$$\iint_{\text{triangolo}} x \, dx \, dy = \int_0^1 dx \int_0^{\sqrt{3}x} dy \cdot x = \sqrt{3} \int_0^1 x^2 dx = \text{banale}$$

↑
normale
asse x

4° modo Polari dirette !!

$$0 \leq \theta \leq \frac{\pi}{3}$$



Fissato θ , dove varia p ?

$$\frac{1}{\cos \theta} \leq p \leq 2$$

$$x=1 \leadsto p \cos \theta = 1 \leadsto p = \frac{1}{\cos \theta}$$

Morale: in coord. polari l'insieme è "normale risp. a θ "

$$\iint_A x \, dx \, dy = \int_0^{\frac{\pi}{3}} d\theta \int_{\frac{1}{\cos \theta}}^2 dp \underbrace{\frac{p \cos \theta}{x}}_x p$$

$$= \int_0^{\frac{\pi}{3}} d\theta \cos \theta \left[\frac{1}{3} p^3 \right]_{p=\frac{1}{\cos \theta}}^{p=2}$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{3}} \left(8 - \frac{1}{\cos^3 \theta} \right) \cos \theta d\theta = \text{banale}$$