

TECNICHE DI CALCOLO

- Formula di riduzione sui rettangoli
- Formula di riduzione sugli insiemi NORMALI

Morale: un integrale doppio = 2 integrali singoli

RETTANGOLI

Sia $A = [a, b] \times [c, d]$

Sia $f: A \rightarrow \mathbb{R}$ decante.

Allora

$$\iint_A f(x, y) dx dy = \int_a^b \left\{ \int_c^d f(x, y) dy \right\} dx$$

$$\int_a^b dx \int_c^d dy f(x, y)$$

$$= \int_c^d \left\{ \int_a^b f(x, y) dx \right\} dy$$

Oss. La formula di riduzione è l'equivalente del DOUBLE COUNTING della combinatoria

Supponiamo di dover sommare una tabella rett. di numeri, ad esempio gli elementi di una matrice

$$\begin{array}{ccccc} * & * & * & * & * \rightarrow r_1 \\ * & * & * & * & * \rightarrow r_2 \\ * & * & * & * & * \rightarrow r_3 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{array}$$

$$\int dy \int dx f$$



$$r_1 + r_2 + r_3 = c_1 + c_2 + c_3 + c_4 + c_5$$

= somma di tutti i numeri

$$\int dx \int dy f$$



Esempio 1 $A = [0, 1] \times [2, 3]$

$$\iint_A (x^2y + y^2) dx dy = \int_0^1 dx \underbrace{\int_2^3 dy (x^2y + y^2)}_{\text{considero } x \text{ una costante e integro in } y}$$

$$= \int_0^1 dx \left[\frac{1}{2} x^2 y^2 + \frac{1}{3} y^3 \right]_{y=2}^{y=3}$$

↑
primitiva risp. ad y

$$= \int_0^1 dx \left\{ \frac{9}{2} x^2 + 9 - 2x^2 - \frac{8}{3} \right\}$$

$$= \int_0^1 dx \left\{ \frac{5}{2} x^2 + \frac{19}{3} \right\} = \left[\frac{5}{6} x^3 + \frac{19}{3} x \right]_{x=0}^{x=1}$$

↑
primitiva risp.
a x

$$= \frac{5}{6} + \frac{19}{3} = \frac{43}{6}$$

Nell'altro modo:

$$\iint_A (x^2y + y^2) dx dy = \int_2^3 dy \int_0^1 dx (x^2y + y^2)$$

$$= \int_2^3 dy \left[\frac{1}{3} x^3 y + y^2 x \right]_{x=0}^{x=1}$$

↑
primitiva in x

$$= \int_2^3 dy \left\{ \frac{1}{3} y + y^2 \right\} =$$

$$\left[\frac{1}{6} y^2 + \frac{1}{3} y^3 \right]_{y=2}^{y=3} = \frac{9}{6} + 9 - \frac{4}{6} - \frac{8}{3}$$

↑
primitiva in y

$$= \frac{5}{6} + \frac{19}{3} = \text{come prima } \ddot{\smile}$$

Un minimo di scorciatoie

$$\iint_A (x^2y + y^2) dx dy = \iint_A x^2y dx dy + \iint_A y^2 dx dy$$

$$\begin{aligned}\iint_A x^2y dx dy &= \int_0^1 dx \int_2^3 x^2y dy = \int_0^1 x^2 dx \int_2^3 y dy \\ &= \left[\frac{x^3}{3} \right]_{x=0}^{x=1} \left[\frac{y^2}{2} \right]_{y=2}^{y=3} \\ &= \frac{1}{3} \cdot \frac{5}{2} = \frac{5}{6}\end{aligned}$$

$$\iint_A y^2 dx dy = \int_0^1 dx \int_2^3 y^2 dy = \int_0^1 dx \left[\frac{y^3}{3} \right]_{y=2}^{y=3} = \int_0^1 \frac{19}{3} dx = \frac{19}{3} \cdot 1$$

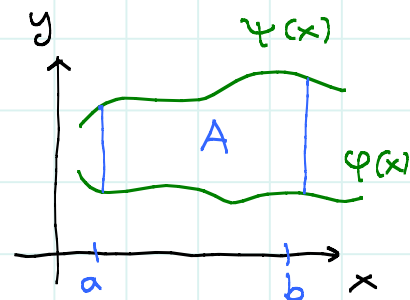
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lungh.
intervallo

INSIEMI NORMALI Un insieme $A \subseteq \mathbb{R}^2$ si dice

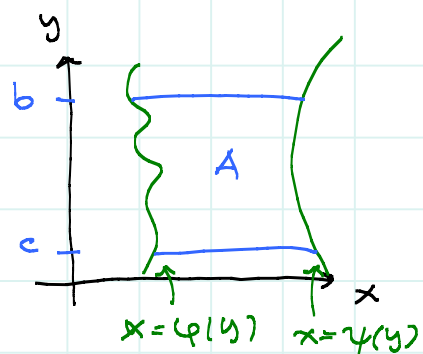
- normale rispetto all'asse x se è del tipo

$$A = \{ (x, y) \in \mathbb{R}^2 : a \leq x \leq b, \varphi(x) \leq y \leq \psi(x) \}$$



- normale rispetto all'asse y se è del tipo

$$A = \{ (x, y) \in \mathbb{R}^2 : c \leq y \leq d, \varphi(y) \leq x \leq \psi(y) \}$$

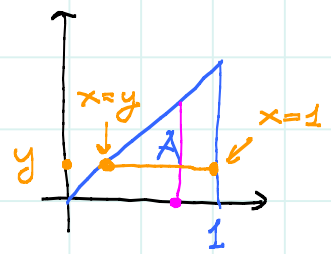


Formule di riduzione

Asse x :
$$\iint_A f(x, y) dx dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x, y) dy$$

Asse y :
$$\iint_A f(x, y) dx dy = \int_c^d dy \int_{\varphi(y)}^{\psi(y)} f(x, y) dx$$

Esempio $A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$



$$\iint_A x^2 y^3 dx dy$$

1° modo Normale axe x

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\}$$

$$\begin{aligned} \iint_A x^2 y^3 dx dy &= \int_0^1 dx \int_0^x dy x^2 y^3 = \int_0^1 x^2 dx \int_0^x y^3 dy \\ &= \int_0^1 x^2 dx \left[\frac{1}{4} y^4 \right]_{y=0}^{y=x} = \frac{1}{4} \int_0^1 x^6 dx = \boxed{\frac{1}{28}} \end{aligned}$$

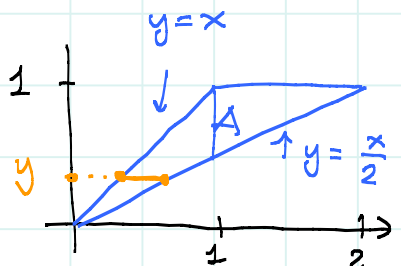
2° modo Normale axe y

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1, y \leq x \leq 1\}$$

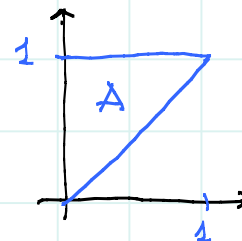
$$\begin{aligned} \iint_A x^2 y^3 dx dy &= \int_0^1 dy \int_y^1 dx x^2 y^3 = \int_0^1 y^3 dy \int_y^1 x^2 dx \\ &= \int_0^1 y^3 dy \left[\frac{1}{3} x^3 \right]_{x=y}^{x=1} = \frac{1}{3} \int_0^1 y^3 (1 - y^3) dy \\ &= \frac{1}{3} \int_0^1 (y^3 - y^6) dy = \frac{1}{3} \left[\frac{1}{4} y^4 - \frac{1}{7} y^7 \right]_{y=0}^{y=1} \\ &= \frac{1}{3} \left(\frac{1}{4} - \frac{1}{7} \right) = \frac{1}{3} \frac{3}{28} = \boxed{\frac{1}{28}} \quad \therefore \end{aligned}$$

Esempio 2

$$\begin{aligned} \iint_A f dx dy &= \int_0^1 dx \int_{\frac{x}{2}}^x dy f + \int_1^2 dx \int_{\frac{x}{2}}^1 dy f \\ &= \int_0^1 dy \int_y^{2y} dx f \end{aligned}$$



Esempio 3 $\iint_A e^{-y^2} dx dy$



1° modo : asse x

$$\int_0^1 dx \int_x^1 dy e^{-y^2} = \text{☹️} \text{ non so fare la primitiva in } y$$

2° modo : asse y

$$\int_0^1 dy \int_0^y dx e^{-y^2} = \int_0^1 e^{-y^2} dy \underbrace{\int_0^y 1 \cdot dx}_{\text{lunghezza.} \text{caten.}} = \int_0^1 y e^{-y^2} dy$$

e ora so fare la primitiva ☺

Prima mi sono arreso troppo presto

$$\begin{aligned} \int_0^1 dx \int_x^1 dy e^{-y^2} &= \int_0^1 dx \left[F(y) \right]_{y=x}^{y=1} \\ &\quad \uparrow \\ &\quad \text{una primitiva, ad} \\ &\quad \text{esempio quella che si annulla in 1} \\ &= \int_0^1 dx \left\{ F(1) - F(x) \right\} = - \int_0^1 F(x) dx \end{aligned}$$

ma ora integro per parti e la F se ne va ☺
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