

Cambi di variabile negli integrali tripli = esattamente come nei doppi
(solo che la matrice jacobiana è 3×3).

Casi speciali: $\rightarrow$ coord. cilindriche
 $\rightarrow$ coord. sferiche

COORD. CILINDRICHE

ρ, θ, z

$\uparrow$ polari corrispondenti a (x, y)

Formule di passaggio

$$x = \rho \cos \theta \quad y = \rho \sin \theta \quad z = z$$

In questo caso

$$J(\rho, \theta, z) = \rho$$

COORD. SFERICHE

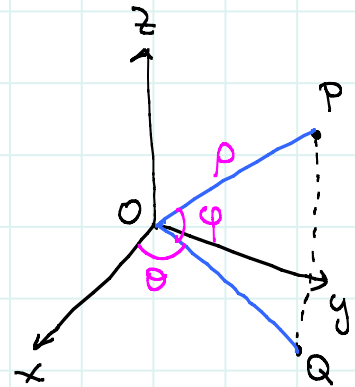
Versione geografica

ρ, θ, φ

$\uparrow$ $\uparrow$ latitudine
longitudine

$$OQ = OP \cdot \cos \varphi = \rho \cos \varphi$$

$$PQ = OP \cdot \sin \varphi = \rho \sin \varphi$$



Formule di passaggio:

$$x = \rho \cos \varphi \cos \theta$$

$$y = \rho \cos \varphi \sin \theta$$

$$z = \rho \sin \varphi$$

In queste notazioni

$$J(\rho, \theta, \varphi) = \rho^2 \cos \varphi$$

Esempio 1

$$\iiint_{S^+} (x^2 + y^2) dx dy dz$$

dove $S^+ := \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0 \}$

1° modo Per colonne

$$S^+ := \{ (x, y, z) \in \mathbb{R}^3 : \underbrace{x^2 + y^2 \leq 1}_{\text{base } \Omega}, 0 \leq z \leq \sqrt{1 - x^2 - y^2} \}$$

$$\iiint_{S^+} (x^2 + y^2) dx dy dz = \iint_{x^2 + y^2 \leq 1} dx dy (x^2 + y^2) \underbrace{\sqrt{1 - x^2 - y^2}}_{\text{lunghezza colonna}}$$

$$= \int_0^1 \rho d\rho \int_0^{2\pi} d\theta \rho^2 \sqrt{1 - \rho^2} \cdot \rho = 2\pi \int_0^1 \rho^3 \sqrt{1 - \rho^2} d\rho$$

= si fa per parti

$\rho^2 \cdot \underbrace{2\rho \sqrt{1 - \rho^2}}_{\substack{\uparrow \text{ si integra bene} \\ \text{si deriva bene}}}$

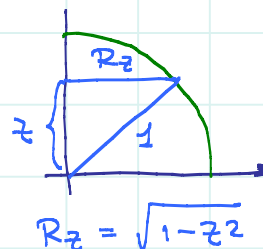
2° modo Per sezioni, che è equivalente alle cilindriche

$$\begin{aligned} \iiint_{S^+} (x^2 + y^2) dx dy dz &= \int_0^1 dz \iint_{S_z} (x^2 + y^2) dx dy \\ &= \int_0^1 dz \int_0^{R_z} \rho d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \rho^2 \cdot \rho \uparrow \text{ pag} \end{aligned}$$

$$= 2\pi \int_0^1 dz \int_0^{R_z} \rho^3 d\rho$$

$$= \frac{\pi}{2} \int_0^1 R_z^4 dz$$

$$= \frac{\pi}{2} \int_0^1 (1 - z^2)^2 dz = \text{si fa ancora meglio.}$$



Una descrizione di S_+ in cilindriche è

$$S_+ := \{ (p, \theta, z) \in \mathbb{R}^3 : 0 \leq z \leq 1, \underbrace{0 \leq p \leq \sqrt{1-z^2}, 0 \leq \theta \leq 2\pi}_{\text{Descrizione della sezione di prima}} \}$$

Mi raccomando evitare

$$\underbrace{0 \leq z \leq 1, 0 \leq p \leq 1, 0 \leq \theta \leq 2\pi}_{\text{cilindro}}$$

Si poteva anche scrivere

$$S_+ := \{ (p, \theta, z) \in \mathbb{R}^3 : \underbrace{0 \leq p \leq 1, 0 \leq \theta \leq 2\pi}_{\text{base}}, \underbrace{0 \leq z \leq \sqrt{1-p^2}}_{\text{colonna}} \}$$

3° modo Sferiche dei geografi

$$S_+ := \{ (p, \theta, \varphi) \in \mathbb{R}^3 : 0 \leq p \leq 1, 0 \leq \theta \leq 2\pi, \underbrace{0 \leq \varphi \leq \frac{\pi}{2}}_{\text{latitudine}} \}$$

$$\begin{aligned} \iiint_{S_+} (x^2 + y^2) dx dy dz &= \int_0^1 dp \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \underbrace{p^2 \cos^2 \varphi}_{x^2 + y^2} p^2 \cos \varphi \\ &= 2\pi \underbrace{\int_0^1 p^4 dp}_{\frac{1}{5}} \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta = \frac{1}{5} \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta \end{aligned}$$

Analisi 1 $\int \cos^3 \theta d\theta = \int \cos^2 \theta \cdot \cos \theta d\theta$

$$= \int (1 - \sin^2 \theta) \cos \theta d\theta = \int (1 - y^2) dy$$

$\uparrow$
 $y = \sin \theta$
 $dy = \cos \theta d\theta$

$$= y - \frac{1}{3} y^3 = \sin \theta - \frac{1}{3} \sin^3 \theta$$

4° modo Mi faccio furbo

$$\iiint_{S+} (x^2+y^2) dx dy dz = \frac{1}{2} \iiint_S (x^2+y^2) dx dy dz$$

↑
tutta la sfera

↑
per
simmetria

$$= \frac{1}{2} \frac{2}{3} \iiint_S (x^2+y^2+z^2) dx dy dz$$

$$= \frac{1}{3} \int_0^1 \rho d\rho \int_0^{2\pi} d\theta \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \rho^2 \rho^2 \cos \varphi$$

$$= \frac{1}{3} \underbrace{\int_0^1 \rho^4 d\rho}_{\frac{1}{5}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi}_{2} = \frac{4\pi}{15}$$

Oss. È collegato al momento di inerzia risp. all'asse z.

Esempio 2 S = sfera con centro in (1, 2, 3) e raggio 2

$$\iiint_S x dx dy dz$$

Scrivo $S = \{ (x, y, z) \in \mathbb{R}^3 : \underbrace{(x-1)^2}_{u^2} + \underbrace{(y-2)^2}_{v^2} + \underbrace{(z-3)^2}_{w^2} \leq 4 \}$

$$\iiint_S x dx dy dz = \iiint_{u^2+v^2+w^2 \leq 4} \underbrace{(u+1)}_x \underbrace{1}_{\uparrow \text{traslazione}} du dv dw = \frac{4}{3} \pi \cdot 8 = \frac{32\pi}{3}$$

$$\iiint_{u^2+v^2+w^2 \leq 4} u du dv dw = 0 \text{ per simmetria!}$$

$$\iiint_{\dots} 1 \dots = \text{Volume sfera}$$

Esempio 3

$$A = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x \geq 1 \}$$

$$\iiint_A x \, dx \, dy \, dz$$

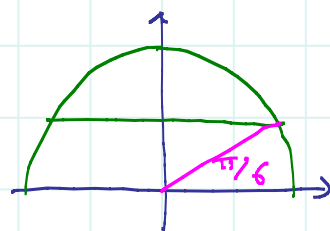
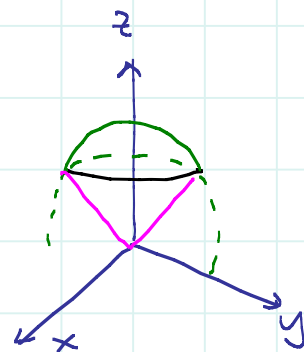
Giro le variabili e calcolo

$$\iiint_B z \, dx \, dy \, dz \quad \text{dove}$$

$$B = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, z \geq 1 \}$$

Per sezioni

$$\begin{aligned} \iiint_B z \, dx \, dy \, dz &= \int_1^2 dz \cdot z \cdot \pi R_z^2 \\ &= \pi \int_1^2 dz \, z (4 - z^2) \end{aligned}$$



Come si scriverebbe B in coord. sferiche?

$$0 \leq \theta \leq 2\pi$$

$$\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{2}$$

$$0 \leq \rho \leq 2$$

cono + gelato

$$\frac{1}{\sin \varphi} \leq \rho \leq 2$$

$$z \geq 1$$

$$\rho \sin \varphi \geq 1$$

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