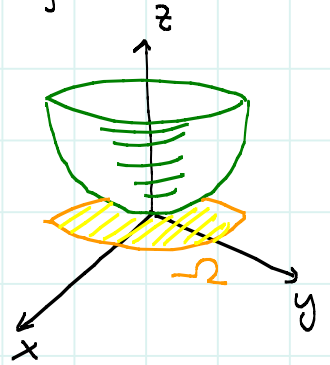


Esempio 1 Ciotola $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z \leq 2\}$

$$\iiint_C z \, dx \, dy \, dz$$

1° modo PER COLONNE



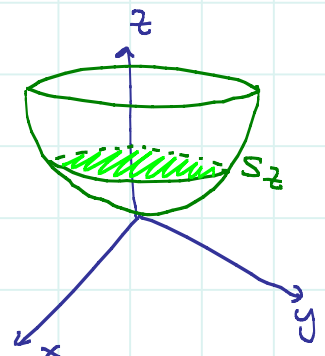
$$C = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in \Omega, \underbrace{x^2 + y^2 \leq z \leq 2}_{\text{colonna}}\}$$

dove $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$

$$\begin{aligned} \iiint_C z \, dx \, dy \, dz &= \iint_{\Omega} dx \, dy \int_{x^2+y^2}^2 dz \, z \\ &= \iint_{\Omega} dx \, dy \left[\frac{1}{2} z^2 \right]_{z=x^2+y^2}^{z=2} \\ &= \frac{1}{2} \iint_{\Omega} [4 - (x^2+y^2)^2] \, dx \, dy \\ &= \frac{1}{2} \int_0^{\sqrt{2}} \rho \, d\rho \int_0^{2\pi} d\theta [4 - \rho^4] \quad \rho \uparrow \text{rag.} \\ &= \pi \int_0^{\sqrt{2}} (4\rho - \rho^5) \, d\rho \\ &= \pi \left[2\rho^2 - \frac{1}{6}\rho^6 \right]_{\rho=0}^{\rho=\sqrt{2}} = \pi \left(4 - \frac{4}{3} \right) = \frac{8}{3}\pi \end{aligned}$$

2° modo: per SEZIONI

$$\begin{aligned} \iiint_C z \, dx \, dy \, dz &= \int_0^2 dz \iint_{S_z} z \, dx \, dy \\ &= \int_0^2 z \, dz \iint_{S_z} 1 \, dx \, dy \end{aligned}$$

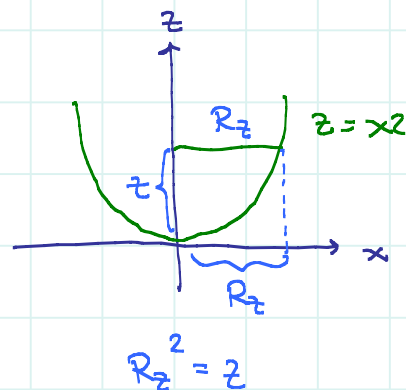


$$= \int_0^2 z \cdot \text{Area}(S_z) dz$$

$$= \pi \int_0^2 z \cdot R_z^2 dz$$

↑
raggio S_z

$$= \pi \int_0^2 z^2 dz = \frac{\pi}{3} [z^3]_{z=0}^{z=2} = \frac{8}{3} \pi \quad \text{😊}$$



Fosse stato $\iiint_C x dx dy dz = 0$ (simmetria)

→ Se faccio la trasformazione $(x, y, z) \rightarrow (-x, y, z)$

l'insieme C va in se stesso

→ L'integrand cambia segno $f(-x, y, z) = -f(x, y, z)$.

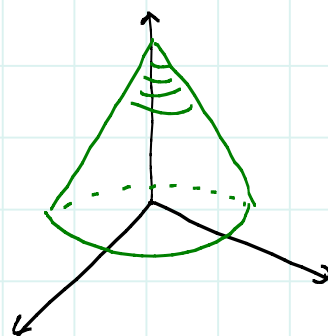
Analisi 1 Se $f(-x) = f(x)$, allora $\int_{-a}^a f(x) dx = 0$

Se $f(-x) = -f(x)$, allora $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

Esempio 2 C = cono come in figura

$r = 2$

altezza $z \in [0, 4]$



$$\iiint_C (z^2 + xy) dx dy dz$$

Intanto $\iiint_C xy dx dy dz = 0$ per la stessa simmetria di prima
(sia risp. ad x , sia risp. ad y)

1° modo Per sezioni

$$\iiint_C z^2 dx dy dz = \int_0^4 dz z^2 \iint_{S_z} dx dy = \int_0^4 dz z^2 \cdot \text{Area}(S_z)$$

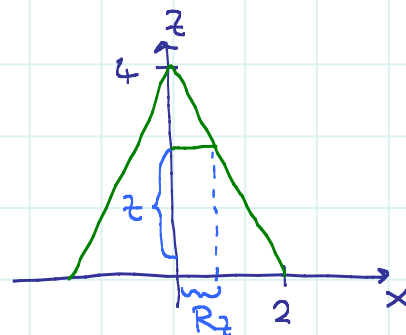
↑
cerchio di raggio R_z

$$= \pi \int_0^4 z^2 \left(\frac{4-z}{2} \right)^2 dz = \text{si fa}$$

$$z = 4 - 2x$$

$$z = 4 - 2R_2$$

$$R_2 = \frac{4-z}{2}$$



2° modo Per colonne

$$C = \{ (x, y, z) \in \mathbb{R}^3 : \underbrace{x^2 + y^2 \leq 4}_{\text{base } \Omega}, 0 \leq z \leq 4 - 2\sqrt{x^2 + y^2} \}$$

Senza l'equazione del cono $z = 4 - 2\rho = 4 - 2\sqrt{x^2 + y^2}$

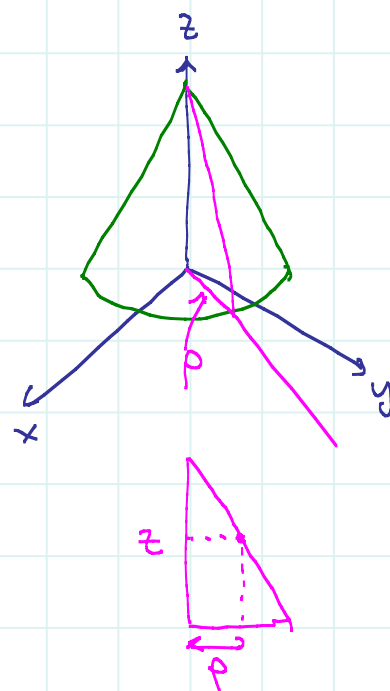
Volendo si arrivava anche dalla sezione

$$x^2 + y^2 \leq R_z^2 = \left(\frac{4-z}{2} \right)^2$$

Quindi vale il seguente di = quando

$$x^2 + y^2 = \left(\frac{4-z}{2} \right)^2$$

$$\sqrt{x^2 + y^2} = \frac{4-z}{2} \rightarrow \text{ricavo } z$$



$$\iiint_C z^2 dx dy dz = \iint_{x^2 + y^2 \leq 4} dx dy \int_0^{4 - 2\sqrt{x^2 + y^2}} z^2 dz$$

$$= \frac{1}{3} \iint_{x^2 + y^2 \leq 4} (4 - 2\sqrt{x^2 + y^2})^3 dx dy$$

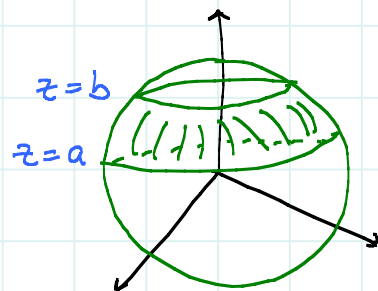
$$= \frac{1}{3} \int_0^2 d\rho \int_0^{2\pi} d\theta (4 - 2\rho)^3 \rho = \text{si fa e forse viene come l'altro.}$$

Esempio 3 Volume del segmento sferico

$$S_{a,b}(R) := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq R^2, a \leq z \leq b\}$$

Fatto generale Dato $S \subseteq \mathbb{R}^3$ vale che

$$\text{Vol}(S) = \iiint_S 1 \, dx \, dy \, dz \quad (\text{tecnicamente \textit{e'} la def. di Volume})$$



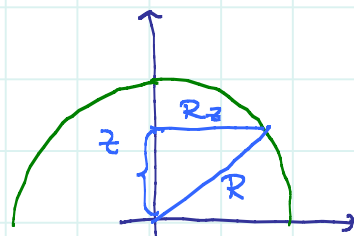
Descritto per colonne \textit{e} onibile. Quindi proviamo per sezioni.

$$\iiint_S 1 \, dx \, dy \, dz = \int_a^b dz \, \text{Area}(S_z) =$$

$$= \pi \int_a^b R_z^2 \, dz$$

$$= \pi \int_a^b (R^2 - z^2) \, dz$$

$$= \pi \left\{ R^2(b-a) - \frac{1}{3}(b^3 - a^3) \right\}$$



Caso speciale Se $a = -R$ e $b = R$ viene $\pi \left\{ R^2 \cdot 2R - \frac{1}{3} 2R^3 \right\}$
 $= \pi R^3 \left(2 - \frac{2}{3} \right) = \frac{4}{3} \pi R^3$

Esempio 4 $T := \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1\}$

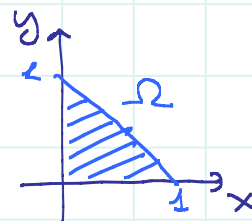
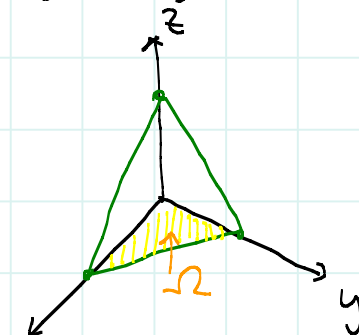
$$\iiint_T y \, dx \, dy \, dz$$

1° modo Per colonne

$$T = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega, 0 \leq z \leq 1-x-y\}$$

dove

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$$



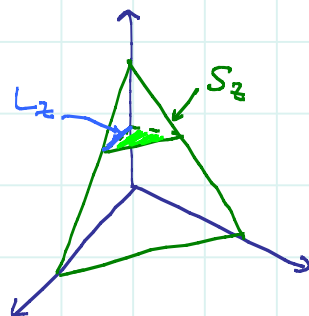
Di conseguenza

$$\begin{aligned}\iint_T y \, dx \, dy \, dz &= \iint_{\Omega} dx \, dy \, y \int_0^{1-x-y} 1 \, dz \\ &= \int_0^1 dx \int_0^{1-x} y \, dy \int_0^{1-x-y} 1 \, dz \\ &= \int_0^1 dx \int_0^{1-x} y (1-x-y) \, dy = \text{si fa}\end{aligned}$$

2° modo Per sezioni

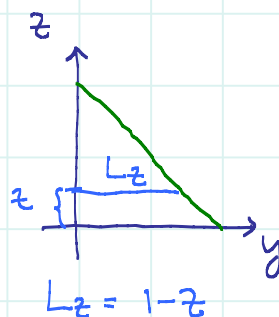
$$\iiint_T y \, dx \, dy \, dz = \int_0^1 dz \iint_{S_z} y \, dx \, dy$$

$\uparrow$
 S_z
 $\uparrow$
triangolo



Se mi faccio furbo

$$\begin{aligned}\iiint_T y \, dx \, dy \, dz &= \iiint_T z \, dx \, dy \, dz \\ &= \int_0^1 z \cdot \text{Area}(S_z) \, dz \\ &= \frac{1}{2} \int_0^1 z \cdot L_z^2 \, dz \\ &\quad \uparrow \\ &\quad \text{cateto di } S_z\end{aligned}$$



$$= \frac{1}{2} \int_0^1 z (1-z)^2 \, dz = \text{si fa facile} \quad \text{☺}$$

— o — o —